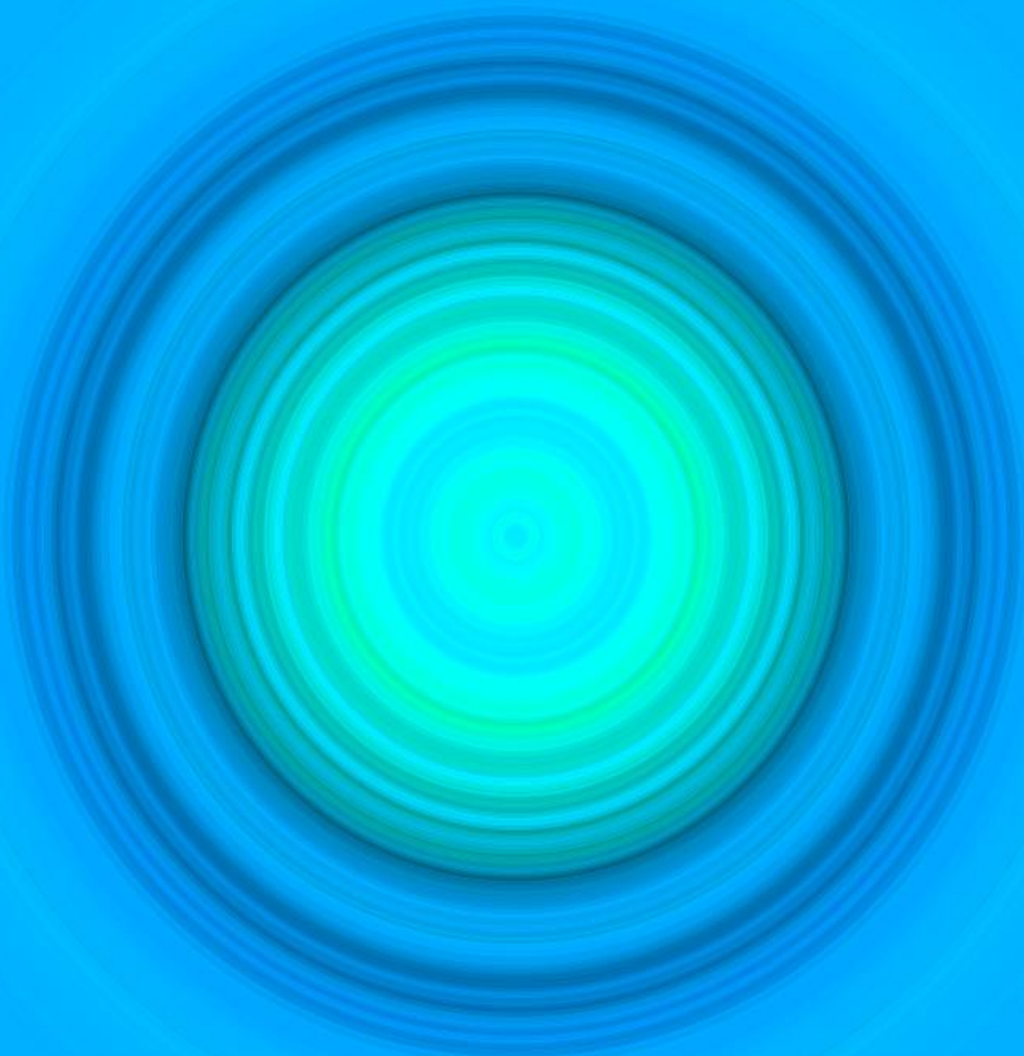
ADVANCED MATHEMATICS
COORDINATE GEOMETRY 1



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To my father George L. B.

COORDINATES GEOMETRY 1

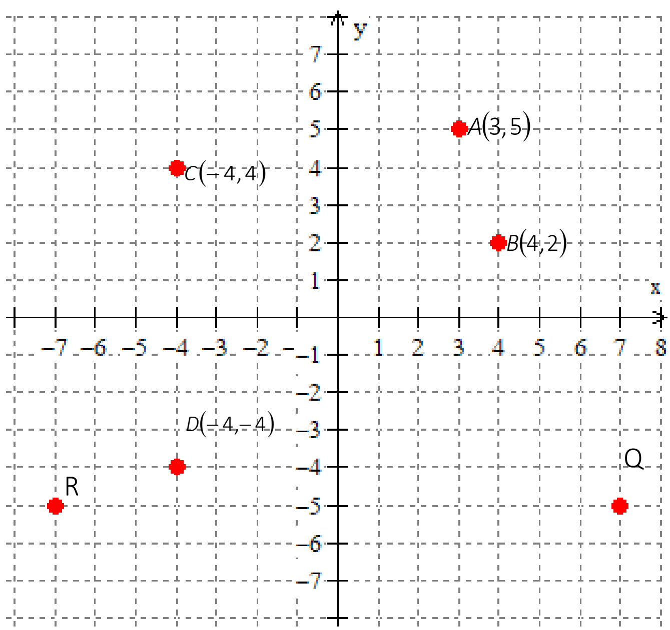
Chapter

4

Coordinates geometry is study of representation of the geometry figure either on two- or three-dimension planes. It is one of the most important and exciting ideas of mathematics. It provides a connection between algebra and geometry through graph of lines and curves. This enables geometric problems to be solved algebraically and provides geometric insights into algebra. There are all sorts of ways that we can find the measurements of lines and angles. We can use rulers to measure lines and protractors to measure angles. In coordinate geometry, we can use graphs and coordinates to find measurements and other useful information about geometrical figures.

1.1. Rectangular coordinates

In coordinates geometry we can specify the position of a point on xy – plane by referring x – axis first and y – axis next.



The coordinates of $A(3,5)$, $B(4,2)$ $C(-4,4)$ and $D(-4,-4)$. The vertical distance is referred ordinate and the horizontal distance is referred to as abscissa. In naming the coordinates, we start with abscissa first and then ordinate, for point A it is $A(3,5)$ to distinguish it from $P(5,3)$. These two points A and P are two different coordinates or points. The system of naming coordinates is what referred to as Rectangular or Cartesian coordinates

Name the coordinate Q and R given in the xy – plane above and also show the points $M(-1,-5)$ and $N(2,-3)$ on xy – plane.

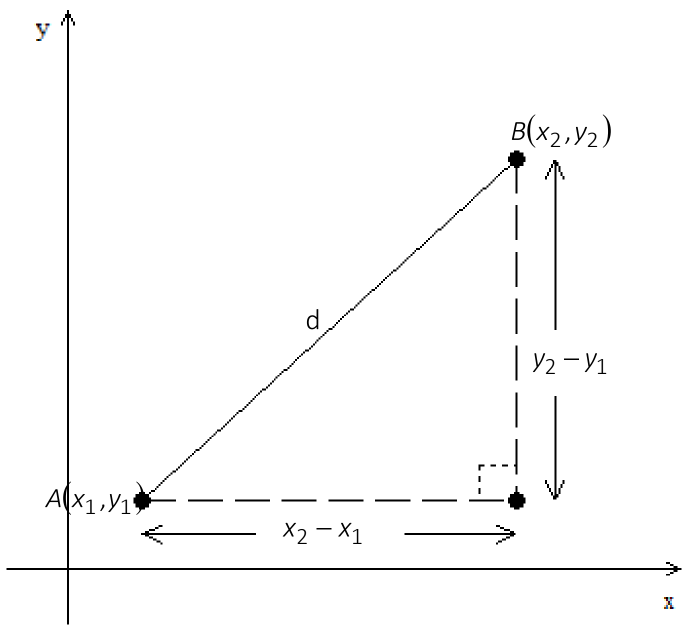
1.2. The distance between two points on the plane

The distance between two points on a plane is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof:

Consider the xy – plane given below



By the Pythagoras theorem, In any right angled $c^2 = a^2 + b^2$ where c is a hypotenuse side, the other sides a and b are adjacent sides.

Therefore, $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \Rightarrow d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example 1

Find the distance between the point $A(3,\ 2)$ and $B(8,\ 14)$

Solution

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$A(3,\ 2)$ and $B(8,\ 14)$

$$\overline{AB}=d=\sqrt{(8-3)^2+(14-2)^2}$$

$$\overline{AB}=\sqrt{5^2+12^2}$$

$\overline{AB}=\sqrt{169}$ = 13 units.

The distance is 13 units

Example 2

The distance between points $P(3,4)$ and $Q(5,m)$ is $\sqrt{20}$ units. Find the possible values of *m.*

Solution

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ thus $d=\sqrt{(3-5)^2+(4-m)^2}$

But given the distance, $d=\sqrt{20}$

Equity the two distance $\sqrt{(-2)^2+(4-m)^2}=\sqrt{20}$

Square both sides to get $4+(4-m)^2=20$

$$(4-m)^2=16$$

Then $\quad m=8$ or $m=0$

Example 3

If the distance between the points $Q(3a,\ b)$ and $P(4a,\ 3b)$ is $3b$, find $a:b$

Solution

Given $Q(3a,b)$, $P(4a,3b)$ and the distance between them is $3b$

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ thus $3b=\sqrt{(3a-4a)^2+(b-3b)^2}$

Squaring both sides $9b^2=a^2+4b^2$

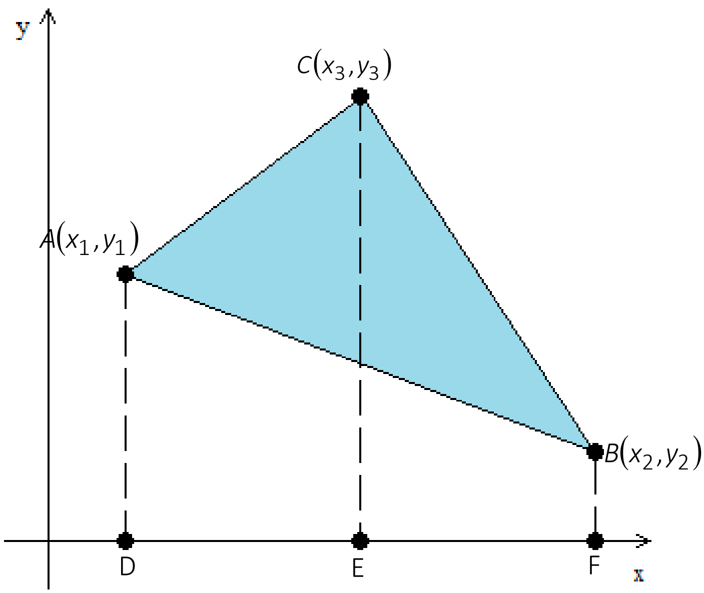
Collecting like terms $5b^2=a^2$

$$\frac{a^2}{b^2}=5 \text{ then } \left(\frac{a}{b}\right)^2=5$$

$$\frac{a}{b}=\sqrt{5} \text{ therefore } a{:}b=\sqrt{5}{:}1$$

1.3. Area of a tringle

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be the vertices of a triangle below



Area of ΔABC = Area of $DACE$ + Area of $ECBF$ – Area of $ABFD$

$$A=\frac{1}{2}h(a+b),\ h=\overline{DE}=x_3-x_1,\ a=y_1 \text{ and } b=y_2$$

$$\text{Area of } DACE=\frac{1}{2}(x_3-x_1)(y_1+y_3)$$

Area of $ECBF = \frac{1}{2}(x_2 - x_3)(y_3 + y_2)$

Area of $ABFD = \frac{1}{2}(x_2 - x_1)(y_1 + y_2)$

Area of $\Delta ABC = \frac{1}{2}[(x_3 - x_1)(y_1 + y_3) + (x_2 - x_3)(y_3 + y_2) - (x_2 - x_1)(y_1 + y_2)]$

Area of $\Delta ABC = \frac{1}{2}\begin{bmatrix} x_3y_1 + x_3y_3 - x_1y_1 - x_1y_3 + x_2y_3 + x_2y_2 \\ -x_3y_3 - x_3y_2 - x_2y_1 - x_2y_2 + x_1y_1 + x_1y_2 \end{bmatrix}$

$$= \frac{1}{2}[x_3y_1 - x_3y_2 + x_1y_2 - x_1y_3 + x_2y_3 - x_2y_1]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)] \qquad **$$

** Above can be expressed as a determinant of 3 by 3 matrix, therefore

Area of $\Delta ABC = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ the absolute signs, ±, ensure that, the area of the triangle is always positive.

Therefore, the area of triangle ABC is given by $\pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Using row reduction, the formula cab be simplified

Area of $\Delta ABC = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix}\begin{matrix} \to R_1 \\ \to R_2 - R_1 \\ \to R_3 - R_1 \end{matrix}$

Area of $\Delta ABC = \pm\frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$

Therefore, the area is now reduced to 2 by 2 determinant, which is easy to evaluate.

1.4. Collinear points

The collinear points are the points, which lies on the same straight line. The points have the same slope. If the area of the triangle is zero (0) the points A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) are said to be collinear points.

Thus, $A = \pm\frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} = 0$ the points are collinear.

Example 4

Find the area of the triangle with vertices A(2, 3), B(1, -2) and C(-1, 4).

Solution

Given vertices A(2, 3), B(1, -2) and C(-1, 4).

By using, area of triangle $= \pm\frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$

$$= \pm\frac{1}{2}\begin{vmatrix} 1-2 & -2-3 \\ -1-2 & 4-3 \end{vmatrix} = \pm\frac{1}{2}\begin{vmatrix} -1 & -5 \\ -3 & 1 \end{vmatrix}$$

$= \pm\frac{1}{2} \times 16 = 8$, thus the area is 8 square units

Example 5

Find the area of the triangle with vertices $P(4,3)$, $Q(8,6)$ and $R(-16,-12)$.

Solution

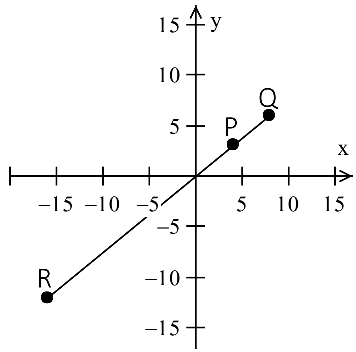
Area $\Delta PQR = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Area of the triangle $= \pm\frac{1}{2}\begin{vmatrix} 4 & 3 & 1 \\ 8 & 6 & 1 \\ -16 & -12 & 1 \end{vmatrix}$

$$= \pm\frac{1}{2}\begin{vmatrix} 8-4 & 6-3 \\ -16-4 & -12-3 \end{vmatrix}$$

$$=\pm\frac{1}{2}\begin{vmatrix} 4 & 3 \\ -20 & -15 \end{vmatrix}=\pm\frac{1}{2}(-60+60)=0$$

The area is zero square units. Geometrical meaning of this is that, the points lies on the same straight line (points are collinear)



See the figure. Line RPQ is a straight line, and all points lies on it, *no enclosed area formed*, that is why the area is zero.

Two cases where area of the tringle is zero.

1. If the points are collinear
2. If the point is repeating

EXERCISE 1 COORDINATE 1

1. Find the distance between the following pair of points

(a) $A(3,\ 8)$ and $B(7,\ -1)$

(b) $A(1,\ 2)$ and $B(5,\ -1)$

(c) $P(9,\ 10)$ and $Q(2,\ -2)$

(d) M(2a, b) and N(3a, 2b)

2. Find the area of the triangle enclosed by the following coordinates

(a) $(-2,-3)$, $(-7,5)$ and $(3,-5)$

(b) $(0,0)$, $(2,0)$ and $(4,5)$

(c) $(2,5)$, $(8,7)$ and $(10,3)$

(d) $(3,5)$, $(6,10)$ and $(0,0)$

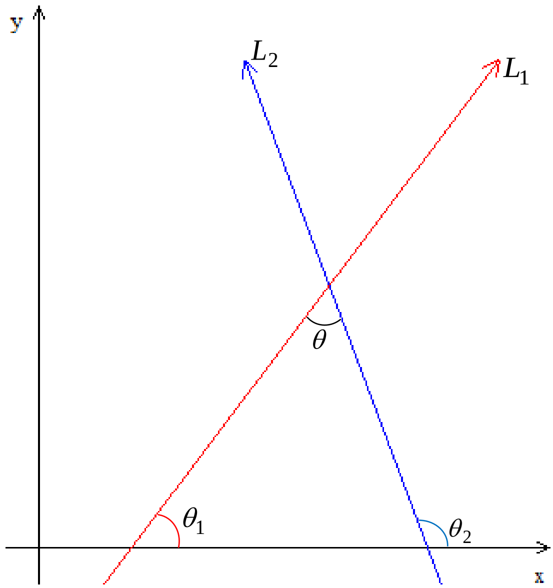
3. Find the relation between x and y if the point (x,y) lie on the line joining the points $(2,3)$ and $(5,4)$.

4. If the point $C(a,\ b)$ is on the line AB where $A(2,\ 4)$ and $B(-1,-3)$ find the equation connecting a and b.

5. If A (1, -3), B (-2, 3) and C (k, 7) are collinear points, what is the value of k?

1.5. Angle between two lines

Let L_1 and L_2 be two given lines, let θ_1 and θ_2 be the angle made by lines L_1 and L_2 with positive $x-$axis.



Thus $\theta_1 + \theta = \theta_2$ then $\theta = \theta_2 - \theta_1$

Using the natural tangent law, slope $= m = \tan\theta$

$$\tan\theta = \tan(\theta_2 - \theta_1)$$

$$\tan(A\text{-}B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

Now $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$

$$\tan\theta = \pm\left(\frac{m_2 - m_1}{1 + m_2 m_1}\right)$$ for acute and obtuse angle

If $\theta = 90°$ the two lines are perpendicular

Therefore, $\tan 90^\circ = \dfrac{\sin 90^\circ}{\cos 90^\circ} = \dfrac{1}{0}$ (not defined)

$$\tan 90^\circ = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$\dfrac{1}{0} = \dfrac{m_2 - m_1}{1 + m_2 m_1}$ therefore $m_2 m_1 = -1$

For perpendicular lines, $m_2 m_1 = -1$ (the product of the slopes of the perpendicular lines is negative one)

If $\theta = 0^\circ$ the two lines are parallel, therefore

$$\tan 0^\circ = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$\dfrac{0}{1} = \dfrac{m_2 - m_1}{1 + m_2 m_1}$ therefore $m_2 - m_1 = 0$ then $m_2 = m_1$

If two lines are parallel, their slopes are equal.

Example 6

Find the angle between lines $x + 3y - 6 = 0$ and $2x + 4y + 1 = 0$

Solution

The angle between lines is given by $\tan\theta = \dfrac{m_2 - m_1}{1 + m_2 m_1}$

$$x + 3y - 6 = 0 \Rightarrow y = -\frac{1}{3}x + 2$$

Then, $m_1 = -\dfrac{1}{3}$

$$2x + 4y + 1 = 0 \Rightarrow y = -\frac{1}{2}x - \frac{1}{4}$$

Then, $m_2 = -\dfrac{1}{2}$

$$\tan\theta = \frac{-\frac{1}{2}-\left(-\frac{1}{3}\right)}{1+\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)} = -\frac{1}{6}\div\frac{7}{6} = -\frac{1}{7}$$

$$\tan\theta = -0.1429$$

$$\theta = \tan^{-1}(-0.1429)$$

$$\theta = 171.9° \text{ or } \theta = 351.9°$$

Example 7

Two parallel lines *AB* and *CD* pass through the points $A(5,0)$ and $C(-5,0)$ respectively. Find the slope of these lines if they meet the line $4x+3y=25$ in points *P* and *Q* such that the distance $\overline{PQ}$ is 5 units.

Solution

For parallel lines $m_1 = m_2$, let the slope of the equations *AB* and *CD* be *m*

The equation of the lines $y = m(x-x_1)+y_1$

Line AB: $y = m(x-5)$ at $A(5,0)$

Line *CD*: $y = m(x+5)$ at $C(-5,0)$

Both these lines intersect with line $4x+3y=25$

Solve simultaneously $\begin{cases} 4x+3y=25 \\ y=m(x-5) \end{cases}$ and $\begin{cases} 4x+3y=25 \\ y=m(x+5) \end{cases}$

Case 1: Solve $\begin{cases} 4x+3y=25 \\ y=m(x-5) \end{cases}$

Substituting $y=m(x-5)$ in $4x+3y=25$

$$4x+3m(x-5)=25 \Rightarrow x(3m+4)=25+15m$$

Therefore, $x = \dfrac{15m+25}{3m+4}$

For y

$$y = m(x-5)$$

$$y = m\left(\frac{15m+25}{3m+4} - 5\right)$$

$$y = m\left(\frac{15m+25-15m-20}{3m+4}\right)$$

$$y = \frac{5m}{3m+4}$$

The coordinate of intersection is $P\left(\frac{15m+25}{3m+4}, \frac{5m}{3m+4}\right)$

Case 2: solve $\begin{cases} 4x+3y=25 \\ y=m(x+5) \end{cases}$

Substituting y: $4x+3m(x+5)=25$

$$4x+3m(x+5)=25 \Rightarrow x(3m+4)=25-15m$$

$$x = \frac{25-15m}{3m+4}$$

$$y = m\left(\frac{25-15m}{3m+4} + 5\right)$$

$$y = m\left(\frac{25-15m+15m+20}{3m+4}\right)$$

$$y = \frac{45m}{3m+4}$$

The point of intersection is $(x,y) = Q\left(\frac{25-15m}{3m+4}, \frac{45m}{3m+4}\right)$

Then the coordinate of P and Q are $P\left(\frac{15m+25}{3m+4},\frac{5m}{3m+4}\right)$ and $Q\left(\frac{25-15m}{3m+4},\frac{45m}{3m+4}\right)$

The distance $\overline{PQ}=5$ (given)

Distance formula $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$$\left(\frac{25-15m}{3m+4}-\frac{25+15m}{3m+4}\right)^2+\left(\frac{45m}{3m+4}-\frac{5m}{3m+4}\right)^2=5^2$$

$$\left(\frac{-30m}{3m+4}\right)^2+\left(\frac{40m}{3m+4}\right)^2=25$$

$$900m^2+1600m^2=25(3m+4)^2$$

$$2500m^2=25(9m^2+24m+16)$$

$$2500m^2=225m^2+600m+400$$

$$2275m^2-600m-400=0$$

Quadratic formula

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$m=\frac{600\pm\sqrt{4000000}}{4550}$$

Then $m=4/7$ or $m=4/13$

The slopes of these lines is $\frac{4}{7}$ or $\frac{4}{13}$

1.6. Combined line

Let $y=m_1x$ and $y=m_2x$ be any two lines which passes the origin.

These two equations can be combined to get $(y-m_1x)(y-m_2x)=0$

$$y^2 - m_2xy - m_1xy + m_1m_2x^2 = 0$$

$$m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$$

Compare the above equation with

$$ax^2 + 2hxy + by^2 = 0 \Rightarrow \frac{a}{b}x^2 + \frac{2hxy}{b} + y^2 = 0$$

By comparing the two equations, $\begin{cases} m_1 + m_2 = -\dfrac{2h}{b} \\ m_1m_2 = \dfrac{a}{b} \end{cases}$

$\tan(\theta - \alpha) = \dfrac{m_1 - m_2}{1 + m_1m_2}$ can be expressed in terms of a, b and h as follows

Let $\theta - \alpha = \beta$

$$\tan(\theta - \alpha) = \tan\beta = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$\tan\beta = \frac{\sqrt{(m_1 - m_2)^2}}{1 + m_1m_2}$$

$$\tan\beta = \frac{\sqrt{m_1^2 - 2m_1m_2 + m_2^2}}{1 + m_1m_2}$$

$$\tan\beta = \frac{\sqrt{(m_1^2 + m_2^2 - 2m_1m_2)}}{1 + m_1m_2}$$

$$\tan\beta = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$

Substitute the corresponding values $\begin{cases} m_1 + m_2 = -\dfrac{2h}{b} \\ m_1m_2 = \dfrac{a}{b} \end{cases}$

$$\tan\beta = \frac{\sqrt{\left(-2h/b\right)^2 - 4\left(a/b\right)}}{1+\left(a/b\right)}$$

$$\tan\beta = \frac{\sqrt{4h^2/b^2 - 4a/b}}{\frac{a+b}{b}}$$

$$\tan\beta = \frac{\left(2\sqrt{h^2-ab}\right)\div b}{(a+b)\div b} = \frac{\left(2\sqrt{h^2-ab}\right)}{(a+b)}$$

$$\tan\beta = \frac{2\sqrt{h^2-ab}}{a+b} \text{ only if } h^2 - ab \geq 0$$

If $a+b=0$, $\tan\beta = \infty$ then $\beta = 90°$. The two lines are perpendicular.

If $h^2 - ab = 0$, $\tan\beta = 0 \Rightarrow \beta = 0°$. The two lines are parallel

Example 8

Find the equations of the lines with equation $2x^2 + 5xy - 12y^2 = 0$ and find the angle between these lines.

Solution

Factorize $2x^2 + 5xy - 12y^2 = (2x-3y)(x+4y) = 0$

$$2x - 3y = 0 \text{ then } y = \frac{2}{3}x$$

Therefore, $m_1 = \frac{2}{3}$

From $x + 4y = 0$, $y = -\frac{1}{4}x$

Therefore, $m_2 = -\frac{1}{4}$

$$\beta = \tan^{-1}\left(\frac{\frac{2}{3} - \left(-\frac{1}{4}\right)}{1 + \left(\frac{2}{3}\right)\left(-\frac{1}{4}\right)}\right)$$

$$\beta = \tan^{-1}(-1.1) = 132.3°$$

$$\beta = \tan^{-1}\left(-\left(\frac{\frac{2}{3} - \left(-\frac{1}{4}\right)}{1 + \left(\frac{2}{3}\right)\left(-\frac{1}{4}\right)}\right)\right)$$

$$\beta = \tan^{-1}(1.1) = 47.7°$$

The angle between the lines are $47.7°$ and $132.3°$

Example 9

Find the value of λ for which $6x^2 - xy + \lambda y^2 = 0$ represents the

(a) Two perpendicular lines
(b) Two parallel lines
(c) Two distinct lines
(d) Two lines inclines at an angle $45°$

Solution

(a) Given $6x^2 - xy + \lambda y^2 = 0$ compare with $ax^2 + 2hxy + by^2 = 0$

Then $a = 6$, $h = -\frac{1}{2}$ and $b = \lambda$

For two perpendicular lines, $a + b = 0$

$$6 + \lambda = 0 \Rightarrow \lambda = -6$$

Therefore $\Rightarrow \lambda = -6$ if the two lines are parallel.

(b) For parallel lines $\sqrt{h^2 - ab} = 0$

$\sqrt{\left(-\frac{1}{2}\right) - 6\lambda} = 0$

$\frac{1}{4} - 6\lambda = 0 \Rightarrow \lambda = \frac{1}{24}$

The λ is $\frac{1}{24}$

(c) For two distinct lines, $\sqrt{h^2 - ab} \geq 0$

$$\sqrt{\left(-\frac{1}{2}\right)^2 - 6\lambda} \geq 0\text{, then } \lambda \geq \frac{1}{24}$$

For two lines to be distinct can be any value of $\lambda \geq \dfrac{1}{24}$

(d) Inclined an angle of $45°$

$$\tan\beta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\tan 45° = \frac{2\sqrt{\left(-\frac{1}{2}\right)^2 - 6\lambda}}{6+\lambda}$$

$$6+\lambda = 2\sqrt{\frac{1}{4} - 6\lambda}$$

$$(6+\lambda)^2 = 4\left(\frac{1}{4} - 6\lambda\right)$$

$$36 + 12\lambda + \lambda^2 = 1 - 24\lambda$$

$$35 + 36\lambda + \lambda^2 = 0$$

$$\lambda^2 + 36\lambda + 35 = 0$$

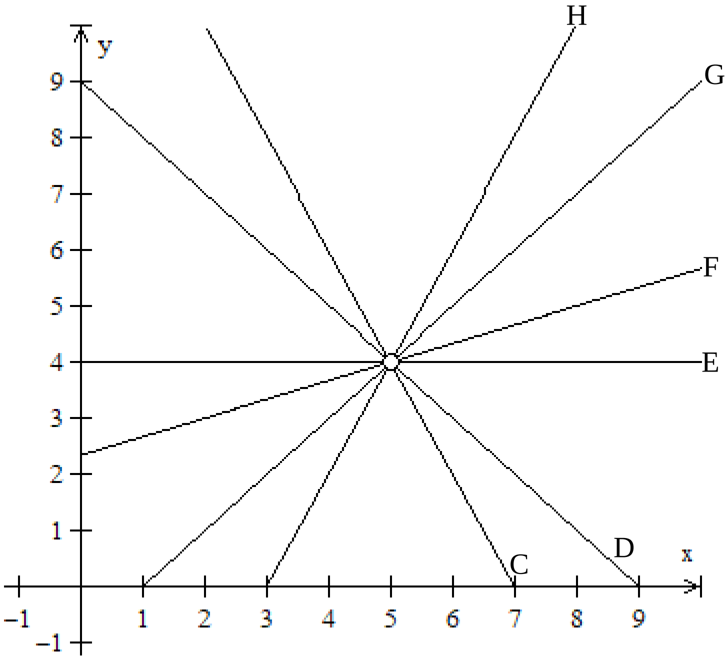
$$(\lambda+1)(\lambda+35) = 0$$

Therefore, $\lambda = -1$ and $\lambda = -35$

1.7. Family lines

These are lines which passes through the common point. From the figure below, all these lines passes through the point $(5, 4)$. These lines are called family lines. This concept of family lines helps to simply the process of finding the equation of the line through the intersection of two or more lines.

Using $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$ where λ the constant, and the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are the two lines which intersects (with the common point)



All the above lines (*C, D, E, F, G, H*) pass through the point (5, 4), they are all family lines

Example 10

Find the equation of the line through the point of intersection of lines $x-3y+6=0$ and line $3x+y-2=0$ and through the point $(4,-2)$

Solution

Given the lines $x-3y+6=0$ and $3x+y-2=0$ the required line passes through the point $(4,-2)$

Find λ : $x-3y+6+\lambda(3x+y-2)=0$ at $(4,-2)$

$$x-3y+6+\lambda(3x+y-2)=0$$

$$16=-8\lambda$$

Substitute $\lambda=-2$

$$x-3y+6-2(3x+y-2)=0$$

The required equation is $-5x-5y+10=0$

Example 11

Find the equation of the line which passes through the point $(3,2)$ and the point of the intersection of the lines $2x+3y-1=0$ and line $3x-4y-6=0$

Solution

Given the lines $2x+3y-1=0$ and $3x-4y-6=0$ the line required pass through the point of intersection and also through the point $(3,2)$

Therefore, $3x-4y-6+\lambda(2x+3y-1)=0$ at $(3,2)$

$$3x-4y-6+\lambda(2x+3y-1)=0$$

$$9-8-6+\lambda(6+6-1)=0$$

$$-5+11\lambda=0 \text{ then } \lambda=\frac{5}{11}$$

$$3x-4y-6+\frac{5}{11}(2x+3y-1)=0$$

$$33x-44y-66+10x+15y-5=0$$

$$43x-29y-71=0$$

The required equation is $43x-29y-71=0$

Example 12

Find the equation of the line through the intersection of lines $3x-2y+14=0$ and $x+y-6=0$ parallel to the line $3x-y+5=0$.

Solution

Given the line passes the intersection of lines $3x-2y+14=0$ and $x+y-6=0$

$$3x-2y+14+\lambda(x+y-6)=0$$

$$3x-2y+14+\lambda x+\lambda y-6\lambda=0$$

$$(3+\lambda)x-(2-\lambda)y-6\lambda+14=0$$

$$(2-\lambda)y=(3+\lambda)x-6\lambda+14$$

$$y=\left(\frac{3+\lambda}{2-\lambda}\right)x-\frac{6\lambda-14}{2-\lambda}$$

$$m_1=\frac{3+\lambda}{2-\lambda}$$

For parallel lines, $m_1=m_2$

The line is parallel to $3x-y+5=0$

$$3x-y+5=0$$

$$y=3x+5$$

$$m_2=3\,,\quad \text{therefore}\quad \frac{3+\lambda}{2-\lambda}=3$$

$$3+\lambda=6-3\lambda$$

$$4\lambda=3\Rightarrow\lambda=\frac{3}{4}$$

Substituting in $3x-2y+14+\lambda(x+y-6)=0$

$$3x-2y+14+\frac{3}{4}(x+y-6)=0$$

$$12x-8y+56+3x+3y-18=0$$

$$15x-5y+38=0$$

The equation of the line required is $15x-5y+38=0$

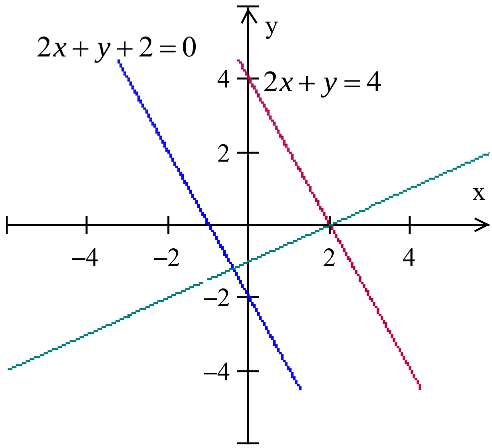
Example 13

Find the perpendicular distance between two parallel straight lines $2x+y=4$ and $2x+y+2=0$

Solution

To find the distance between two parallel lines we should first find any line which is perpendicular to the lines given.

The slope of line $2x+y=4$ is –2, the slope of the perpendicular line is $1/2$, let the new perpendicular line pass through point $(2,0)$ (the assumption



won't affect the answer, any other coordinates can be assumed provided it lies on one the parallel lines)

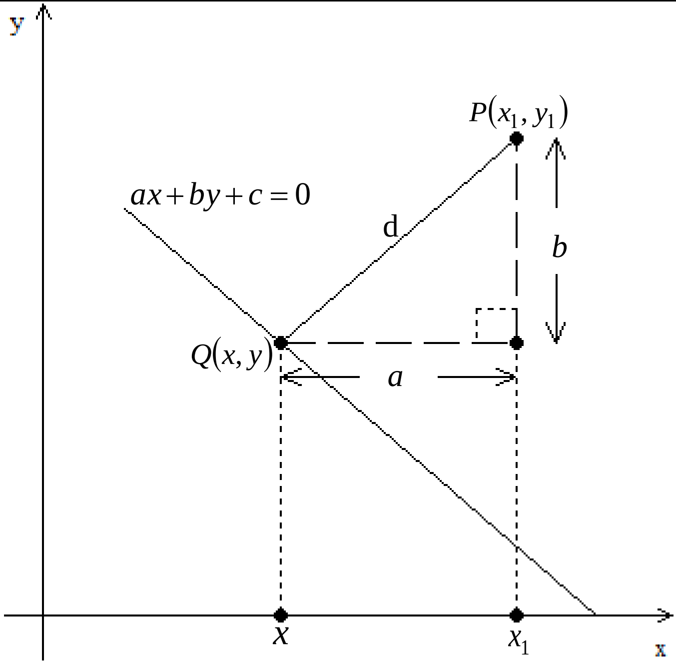
Then, find the perpendicular distance between the point and the other parallel line given. From $d=\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|$

$$d=\left|\frac{2x_1+y_1+2}{\sqrt{2^2+1^2}}\right|=\left|\frac{2(2)+0+2}{\sqrt{5}}\right|=\frac{6}{\sqrt{5}} \text{ units}$$

1.8. Perpendicular distance of a point to a line

The perpendicular distance of a point to a line is given by $d=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

Proof: Consider the xy – plane below



By Pythagoras theorem, $d^2 = a^2 + b^2$ here $d = \overline{PQ}$

$$a = x_1 - x \Rightarrow x = x_1 - a$$

$$b = y_1 - y \Rightarrow y = y_1 - b$$

$$ax + by + c = 0$$

Substituting values of *x* and *y*

$$a(x_1 - a) + b(y_1 - b) + c = 0$$

$$ax_1 - a^2 + by_1 - b^2 + c = 0$$

$ax_1 + by_1 + c = a^2 + b^2$ therefore $d^2 = ax_1 + by_1 + c$

But $d^2 = a^2 + b^2$

$$d = \frac{ax_1 + by_1 + c}{d} = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

Hence, $d = \left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$ given that $a^2 + b^2 \neq 0$

Example 14

Find the perpendicular distance of point $Q(1,3)$ from the line $2x + y + 4 = 0$

Solution

Given line $2x + y + 4 = 0$ and the point $Q(1,3)$

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$d = \left|\frac{2(1) + 3 + 4}{\sqrt{2^2 + 1^2}}\right|$$

$$d = \frac{9}{\sqrt{5}} = \frac{9\sqrt{5}}{5}$$

The distance is $\dfrac{9\sqrt{5}}{5}$ units

Example 15

Show that the perpendicular distance of point $B(3,-2)$ to the line $3x + 4y = 0$

Solution

Given point $B(3,-2)$ and line $3x + 4y = 0$

$$d = \left|\frac{3x + 4y}{\sqrt{3^2 + 4^2}}\right|$$

$$d = \left|\frac{3(3) + 4(-2)}{\sqrt{3^2 + 4^2}}\right| \Rightarrow d = \frac{1}{5}$$

The distance is $1/5$ units.

EXERCISE 2 COORDINATE 1

1. Find the perpendicular distance of the point $C(6,0)$ from the following lines

(a) $4 - 3x + 4y = 0$ (b) $x + y = 2$

(c) $x + 3y - 1 = 0$ (d) $5x - 4 = 0$

(e) $ax + by = 9$

2. Find the perpendicular distance from the point given to the particular line

(a) $(2,3),\ 2x-4+y=0$

(b) $(-1,3),\ 7x-12y+10=0$

(c) $(-10,-2),\ 2x+11=0$

3. Determine the angles between lines

(a) $2x^2+3xy+y^2=0$

(b) $3x^2+10xy+3y^2=0$

(c) $x^2+2xy-3y^2=0$

(d) $4x^2-y^2=0$

4. Find the value of n if $nx^2+2xy+y^2=0$ represent

(a) Two identical lines

(b) Two perpendicular lines

(c) Two lines inclined at 60^{o}

5. Find the condition(s) that the equation $px^2+qxy+ry^2=0$ represent

(a) Two identical lines

(b) Two perpendicular lines

(c) If $2p=2r=q$ what does the equation represent?

6. Show that, if β is the acute angle between the lines that form the line pair $ax^2+2hxy+2y^2=0$ then $\tan\beta=\dfrac{2\sqrt{h^2-2a}}{a+2}$. Deduce the condition that the lines shall be (a) Perpendicular lines (b) Coincident lines.

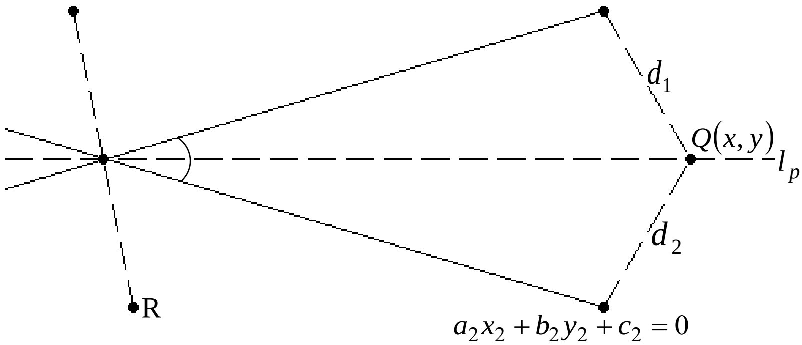
7. Find the angle between the lines $y=\sqrt{3}x+2$ and $\sqrt{3}y=x-4$

1.9. Equation of the perpendicular bisector

Equation of the perpendicular bisector of the angle between two lines can be found by using the formula $\dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

Consider the figure below

P $\quad a_1x_1+b_1y_1+c_1=0$



The equation required is the equation of the line l_p, by using the perpendicular distance formula

Remember also line PR is the perpendicular bisector of the angle between lines $a_1x_1+b_1y_1+c_1=0$ and $a_2x_2+b_2y_2+c_2=0$

$$d=\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|$$

Form the line $a_1x_1+b_1y_1+c_1=0$ to point $Q(x,y)$

$$d_1=\left|\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}\right|$$

From the line $a_2x_2+b_2y_2+c_2=0$ to the point $Q(x,y)$

$$d_2=\left|\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right|$$

Then $d_1=d_2$

The equation of the perpendicular bisector is given by

$$\frac{a_1x+by_1+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

Example 16

Find the equation of the perpendicular bisector of the angles of the lines $3x-4y+2=0$ and $7x+12y-13=0$

Solution

Given lines $3x-4y+2=0$ and $7x+12y-13=0$

$$\frac{3x-4y+2}{\sqrt{3^2+4^2}}=\pm\frac{7x+12y-13}{\sqrt{7^2+12^2}}$$

$$\frac{1}{5}(3x-4y+2)=\pm\frac{1}{13}(7x+12y-13)$$

Case 1: $13(3x-4y+2)=5(7x+12y-13)$

$$39x-52y+26=35x+60y-65$$

$$4x-112y+91=0$$

Case 2: $13(3x-4y+2)=-5(7x+12y-13)$

$$39x-52y+26=-35x-60y+65$$

$$74x+8y-39=0$$

The possible equations are $4x-112y+91=0$ and $74x+8y-39=0$

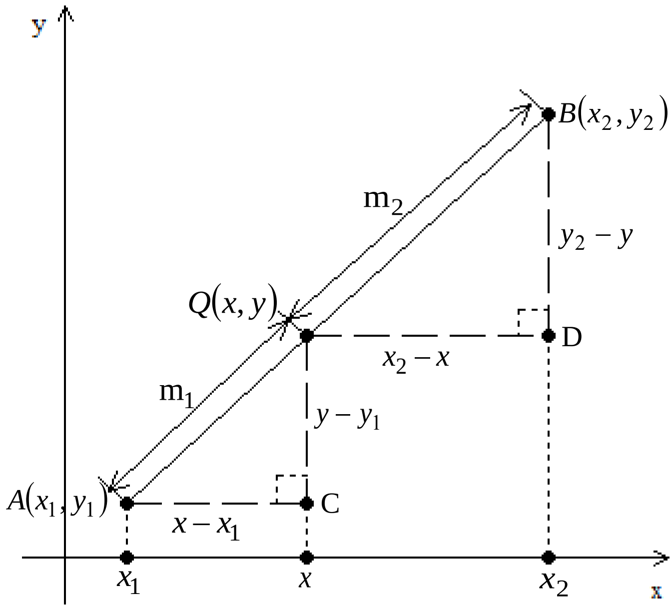
1.10. Division of the line segment (ratio theorem)

A point in any required ratio can divide the line segment internally or externally. In any case, if given the ratio we can find the coordinate of the point that divide the line into the given ratio.

1.11. Internal division of the line

Suppose the point $Q(x,y)$ divides line segment joining point points $A(x_1,y_1)$ and $B(x_2,y_2)$ internally in the ratio $m_1:m_2$

Consider the figure below



From two triangle *ACQ* and *QDB*

Similarity theorem $\dfrac{\overline{AC}}{\overline{QD}} = \dfrac{\overline{CQ}}{\overline{DB}} = \dfrac{\overline{AQ}}{\overline{QB}}$

Then $$\frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y} = \frac{m_1}{m_2}$$

Case1: $$\frac{x - x_1}{x_2 - x} = \frac{m_1}{m_2}$$

$$m_2(x - x_1) = m_1(x_2 - x)$$

$$m_2 x - m_2 x_1 = m_1 x_2 - m_1 x$$

Collecting like terms and solve for x

$$m_2 x + m_1 x = m_1 x_2 + m_2 x_1$$

$$(m_2 + m_1)x = m_1 x_2 + m_2 x_1$$

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$$

$$\frac{y - y_1}{y_2 - y} = \frac{m_1}{m_2}$$

$$m_1(y_2 - y) = m_2(y - y_1)$$

$$m_1 y_2 - m_1 y = m_2 y - m_2 y_1$$

$$(m_2 + m_1)y = m_2 y_1 + m_1 y_2$$

$$y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

Therefore the coordinate $Q(x,y) = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \right)$

When $m_1 = m_2$ *then Q divide the line in two equal parts*

$$Q(x,y) = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \right)$$

$$Q(x,y) = \left(\frac{m_1 x_2 + m_1 x_1}{m_1 + m_1}, \frac{m_1 y_2 + m_1 y_1}{m_1 + m_1} \right)$$

$$Q(x,y) = \left(\frac{m_1(x_2 + x_1)}{2m_1}, \frac{m_1(y_2 + y_1)}{2m_1} \right)$$

Mid – point formula $Q(x,y) = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right)$.

Example 17

Find the coordinates of the point, which divides the line segment joining point, $A(3,4)$ and $B(-4,7)$ in the ratio $3:2$ internally.

Solution

$$Q(x,y)=\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$$

Given $m_1=3$ and $m_2=2$

$(x_1,y_1)=(3,\ 4)$ and $(x_2,y_2)=(-4,\ 7)$

$$Q(x,y)=\left(\frac{3(-4)+2(3)}{3+2},\frac{3(7)+2(4)}{3+2}\right)$$

$$Q(x,y)=\left(\frac{-12+6}{5},\frac{21+8}{5}\right)$$

$$Q(x,y)=\left(-\frac{6}{5},\frac{29}{5}\right)$$

The coordinate required is $\left(-\frac{6}{5},\frac{29}{5}\right)$

Example 18

Find the coordinate of the point that divides the line segment with endpoints $A(3,8)$ and $B(3,2)$ into ratio $2:1$ internally.

Solution

The internal division formula is $Q(x,y)=\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$.

Q is he required coordinate. Given $m_1=2$ and $m_2=1$

$(x_1,y_1)=(3,\ 8)$ and $(x_2,y_2)=(3,\ 2)$

$$Q(x,y)=\left(\frac{2(3)+1(3)}{2+1},\frac{2(2)+1(8)}{2+1}\right)$$

$$Q(x,y)=(3,\ 4)$$

The coordinate is $(3,4)$

Example 19

Find the ratio in which the point $D\left(3,\ \frac{8}{3}\right)$ divides the line segment with endpoints $B(2,\ 3)$ and $C(5,\ 2)$ internally.

Solution

Internal division $Q(x,y)=\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$

$$\left(3,\frac{8}{3}\right)=\left(\frac{m_1(5)+m_2(2)}{m_1+m_2},\frac{m_1(2)+m_2(3)}{m_1+m_2}\right)$$

Compare

Then $3=\frac{5m_1+2m_2}{m_1+m_2}$

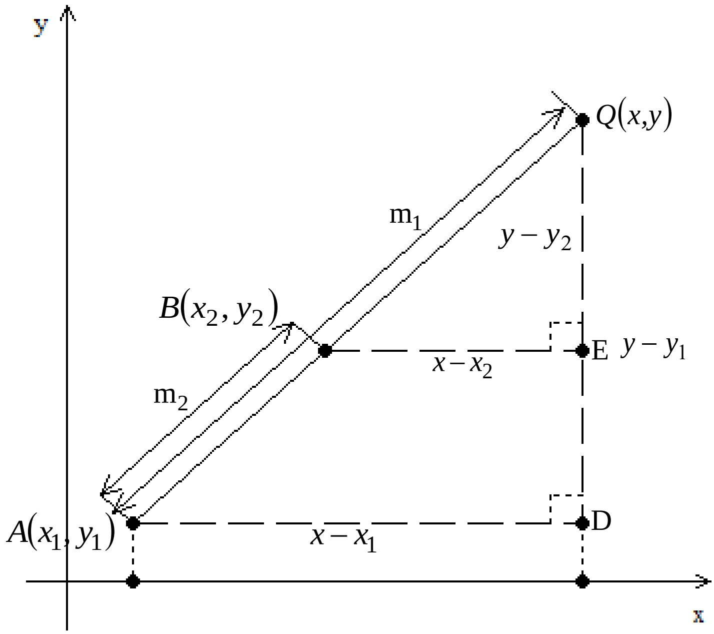
$$3m_1+3m_2=5m_1+2m_2$$

$2m_2=m_1$ therefore $\frac{m_2}{m_1}=2$

$$m_2{:}m_1=2:1$$

Question: Show that the point $\left(3\frac{7}{13},1\frac{2}{3}\right)$ divides the line segment with end-points $(7,3)$ and $(-8,-5)$ in the ratio $3:10$.

1.12. External division of the line



Triangle ADQ and triangle BEQ are similar

By similarity theorem,

$$\frac{\overline{AD}}{\overline{BE}}=\frac{\overline{DQ}}{\overline{EQ}}=\frac{\overline{AQ}}{\overline{BQ}}$$

$$\frac{x-x_1}{x-x_2}=\frac{y-y_1}{y-y_2}=\frac{m_1}{m_2}$$

Case 1: $$\frac{x-x_1}{x-x_2}=\frac{m_1}{m_2}$$

$$m_1(x-x_2)=m_2(x-x_1)$$

$$m_1x-m_1x_2=m_2x-m_2x_1$$

$$m_1x-m_2x=m_1x_2-m_2x_1$$

$$x(m_1-m_2)=m_1x_2-m_2x_1$$

$$x=\frac{m_1x_2-m_2x_1}{m_1-m_2}$$

Case 2:
$$\frac{y-y_1}{y-y_2}=\frac{m_1}{m_2}$$

$$m_1(y-y_2)=m_2(y-y_1)$$

$$m_1y-m_1y_2=m_2y-m_2y_1$$

$$m_1y-m_2y=m_1y_2-m_2y_1$$

$$y(m_1-m_2)=m_1y_2-m_2y_1$$

$$y=\frac{m_1y_2-m_2y_1}{m_1-m_2}$$

The coordinate is $Q(x,y)=\left(\frac{m_1x_2-m_2x_1}{m_1-m_2},\frac{m_1y_2-m_2y_1}{m_1-m_2}\right)$

Example 20

Find the coordinate of the points dividing the line joining the point $(7,-5)$ and $(-2,7)$ externally in the ratio 3:2.

Solution

$$Q(x,y)=\left(\frac{m_1x_2-m_2x_1}{m_1-m_2},\frac{m_1y_2-m_2y_1}{m_1-m_2}\right)$$

$(x_1,y_1)=(7,-2)$ and $(x_2,y_2)=(-2,7)$

$m_1=3$ and $m_2=2$

$$Q(x,y)=\left(\frac{3(-2)-2(7)}{3\text{-}2},\frac{3(7)-2(-5)}{3\text{-}2}\right)$$

$$Q(x,y)=(\text{-}20,\ 31)$$

Example 21

Determine the ratio which make point $C(10, 9)$ to be an external divider of the line segment joining points $A(-2, 1)$ and $B(4, 5)$.

Solution

$$Q(x,y)=\left(\frac{m_1x_2-m_2x_1}{m_1-m_2},\frac{m_1y_2-m_2y_1}{m_1-m_2}\right)$$

$$(10,\ 9)=\left(\frac{m_1(4)-m_2(-2)}{m_1-m_2},\frac{m_1(5)-m_2(1)}{m_1-m_2}\right)$$

$$(10,\ 9)=\left(\frac{4m_1+2m_2}{m_1-m_2},\frac{5m_1-m_2}{m_1-m_2}\right)$$

$$\frac{4m_1+2m_2}{m_1-m_2}=10$$

$$10m_1-10m_2=4m_1+2m_2$$

$$\frac{m_1}{m_2}=2 \text{ hence } m_1:m_2=2:1$$

EXERCISE 3 – COORDINATE 1

1. Find the coordinate of the point which divides the line joining the points $(10, 4)$ and $(3,-3)$ internally into ratio 2:5 and externally into ratio 3:7.
2. Find the coordinate of the point which divides the line segment joining points $(-2,-4)$ and $(-6, 3)$ into ratio internally and 5:3 externally.
3. Derive the internal division formula, and from it deduce the mid-point formula
4. Find the mid-point of the line segment with the end points $P(2, 4)$ and $R(-10,-22)$
5. If the mid-point of the line joining points (a,b) and $(3, 6)$ is $\left(\frac{5}{3},\frac{6}{5}\right)$ find the value of *a* and *b*.
6. Find the equation of the line passing through the point $(3,1)$ and the point of intersection of lines $x+3y-5=0$ and $2x+5y-9=0$

7. Find the value of λ if $2x-y+3-\lambda(x+3y-4)=0$ is parallel to $x-6y+12=0$
8. Find the equation of the line through the point of intersection of the lines $3x+y-7=0$ and $4x-3y-5=0$ and perpendicular with line $x+3y+4=0$

1.13. Locus equation

Locus is a curve or other figure formed by all the points satisfying a particular equation of the relation between coordinate, or by a point, line or surface moving according to mathematically defined condition.

Example 22

Find the equation of the locus of a point R which moves so that it is equidistant from two fixed points X and Y whose coordinates are $(2,5)$ and $(-1,3)$

Solution

Let the point be (x,y), this point is equidistant from the points $(2,5)$ and $(-1,3)$. This means the distance from (x,y) to $(2,5)$ is equal to the distance from (x,y) to the point $(-1,3)$

First distance: $d_1^2=(x-2)^2+(y-5)^2$

$$d_1^2=x^2-4x+y^2-10y+29$$

Second distance $d_2^2=(x+1)^2+(y-3)^2$

$$d_2^2=x^2+2x+y^2-6y+10$$

For equidistant $d_1^2=d_2^2$

$$x^2+y^2-4x-10y+29=x^2+y^2+2x-6y+10$$

$$6x+4y-19=0$$

The required equation is $6x + 4y - 19 = 0$

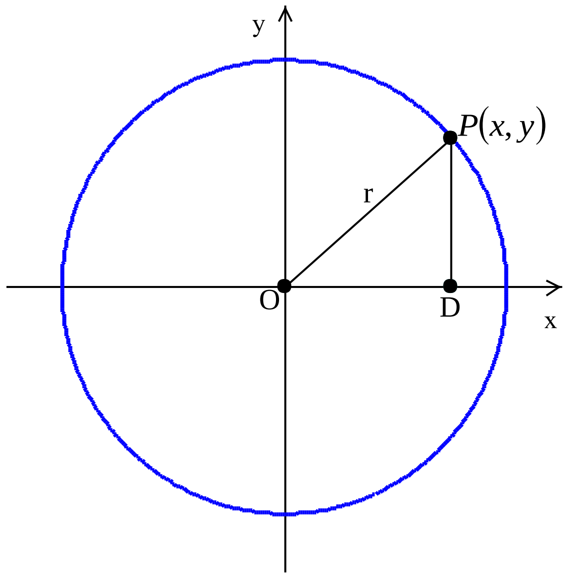
EXERCISE 4 – COORDINATE 1

1. Find the locus of the locus of point P, whose distance from points $A(-1, 2)$ is twice its distance from the origin.
2. Find the locus of the point Q which moves so that it is perpendicular distance from lines $x + \sqrt{3}y - 7 = 0$ is always 7 units
3. Find the locus of the point P which moves so that its distance from point $B(5,-3)$ is always thrice the distance from $C(1,-1)$.
4. Find the locus of the point which moves so that its distance from the point $D(0,-3)$ is equidistant from the line $3y + 2 = 0$

1.14. Circle

The circle is a locus, which moves so that it is distance from a fixed point (center), is always constant (radius).

Below is the circle with center at the origin and the radius of 5 units.



By Pythagoras theorem, $\overline{OD}^2 + \overline{DP}^2 = r^2 \;\Rightarrow\; x^2 + y^2 = r^2$

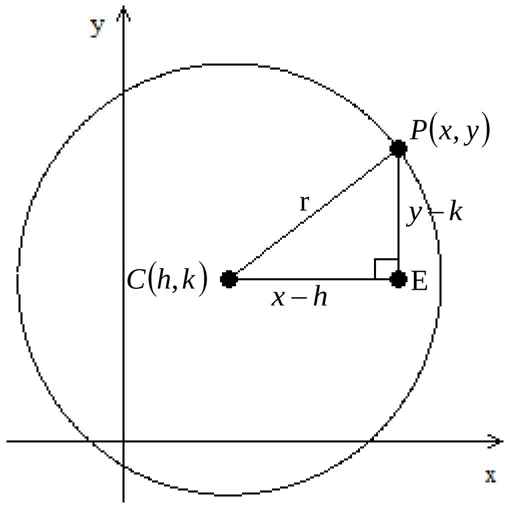
This is a standard equation of the circle, with center $O(0,0)$ and radius *r*.

Properties of equation the circle

1. The coefficients of the terms with x^2 and y^2 are equal
2. There is no term with the product of x and y
3. Always is a polynomial of degree 2

1.15. Transformed or general equation of the circle

If the center of the circle is not at the origin, then the equation of the circle is said to be a transformed equation of the circle or the general equation of the circle. Let assume the center to be $C(h,k)$ and the radius be *r*.



By Pythagoras theorem, $\overline{CE}^2 + \overline{PE}^2 = r^2$

$$(x-h)^2 + (y-k)^2 = r^2$$
$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$
$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

Let $h = -g$ and $k = -f$

$$c = h^2 + k^2 - r^2$$
$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$
$$x^2 + y^2 + 2gx + 2fy + c = 0$$

This is the equation of the circle with center $(-g, -f)$ and radius,

$r = \sqrt{g^2 + f^2 - c}$

Furthermore, we can write the equation as $ax^2 + by^2 + 2gx + 2fy + c = 0$ provided that $a = b \neq 0$

Example 23

Find the coordinate of the center and radius of the circle with equation

(a) $x^2 + y^2 + 5x - 3y + 2 = 0$

(b) $3x^2+3y^2+7x-9y-4=0$

(c) $2x^2+2y^2+10x-5=0$

(d) $x^2+y^2-2x+4y+1=0$

Solution

(a) $x^2+y^2+5x-3y+2=0$ Compare with

$x^2+y^2+2gx+2fy+c=0$

$2g=5 \Rightarrow g=\frac{5}{2}$

$2f=-3 \Rightarrow f=-\frac{3}{2}$ and $c=2$

$C(-g,-f)=\left(-\frac{5}{2},\frac{3}{2}\right)$

$r=\sqrt{g^2+f^2-c}$

$r=\sqrt{\left(-\frac{5}{2}\right)^2+\left(\frac{3}{2}\right)^2-2}$

$r=\sqrt{\frac{13}{2}}=\frac{\sqrt{26}}{2}$

The center is $\left(-\frac{5}{2},\frac{3}{2}\right)$ and radius is $r=\frac{\sqrt{26}}{2}$ units

(b) $3x^2+3y^2+7x-9y-4=0$

Complete the square one side method

$3x^2+3y^2+7x-9y-4=0$

Divide by 3 throughout

$x^2+y^2+\frac{7}{3}x-3y-\frac{4}{3}=0$

$\left(x+\frac{7}{6}\right)^2-\frac{49}{36}+\left(y-\frac{3}{2}\right)^2-\frac{9}{4}-\frac{4}{3}=0$

$\left(x+\frac{7}{6}\right)^2+\left(y-\frac{3}{2}\right)^2-\frac{89}{18}=0$

$$\left(x+\frac{7}{6}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{89}{18}$$

$$\left(x+\frac{7}{6}\right)^2+\left(y-\frac{3}{2}\right)^2=\left(\sqrt{\frac{89}{18}}\right)^2$$

The center of the circle is $C\left(-\frac{7}{6},\frac{3}{2}\right)$

The radius is $r=\sqrt{\frac{89}{18}}$ units

(c) $2x^2+2y^2+10x-5=0$

By completing the square method

$2x^2+2y^2+10x-5=0$

Dividing by 2 throughout

$$x^2+y^2+5x-\frac{5}{2}=0$$

$$\left(x+\frac{5}{2}\right)^2+y^2-\frac{25}{4}-\frac{5}{2}=0$$

$$\left(x+\frac{5}{2}\right)^2+y^2=\frac{35}{4}$$

$$\left(x+\frac{5}{2}\right)^2+y^2=\left(\frac{\sqrt{35}}{2}\right)^2$$

The center is $C\left(-\frac{5}{2},0\right)$ and the radius is $r=\frac{\sqrt{35}}{2}$ units

(d) $x^2+y^2-2x+4y+1=0$

Completing the square

$x^2+y^2-2x+4y+1=0$

$(x-1)^2+(y+2)^2=4$

$(x-1)^2+(y+2)^2=2^2$

The center is $C(1,-2)$ and the radius is $r=2$

EXERCISE 5 – COORDINATE 1

1. Find the coordinate of the center and the radius of the following equation of circles

(a) $x^2+y^2+5x-6y-5-0$

(b) $x^2+y^2-4x-10y+13=0$

(c) $x^2+y^2+2x-6y+6=0$

(d) $4x^2+4y^2+16x-4y-19=0$

(e) $3x^2+3y^2-8x-7=0$

(f) $5x^2+5y^2-8x-10y+15=0$

2. Which among the equations given below are not equation of the circles, and for those circles find the center and the radius.

(a) $x^2-y^2+13x+6y+2=0$

(b) $x^2+2y^2+3x+4y=0$

(c) $10x^2+10y^2-11x+12y-100=0$

(d) $3x^2+4y+3x-5=0$

(e) $x^2+y^2+3xy-8y+4=0$

(f) $2x^2+2y^2-15=0$

1.16. Equation of the circle given three points on the circle

So far we learned circle equation properties and able to determine the center and radius of the given circle equation. Here we will discuss how to find the equation of the circle if you are provided with at least 3 points that lies on the circle.

Note that at any point the equation of the circle is given by $(x-a)^2+(y-b)^2=r^2$ where a and b are the center coordinate and r is the radius of the circle.

If $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are the points given that lies of the circle then the equations $\begin{cases}(x_1-a)^2+(y_1-b)^2=r^2\\(x_2-a)^2+(y_2-b)^2=r^2\\(x_3-a)^2+(y_3-b)^2=r^2\end{cases}$ can be satisfied by all three points.

Example 24

Find the equation of the circle which passes through the points $A(3,1)$, $B(2,6)$ and $C(3,2)$

Solution

Let use the equation $x^2+y^2+2gx+2fy+c=0$

At $A(3,1)$ therefore $6g+2f+c=-10$ (i)

At $B(2,6)$ therefore $8g+12f+c=-40$ (ii)

At $C(3,2)$ therefore $6g+4f+c=-13$ (iii)

Solve simultaneously $\begin{cases}6g+2f+c=-10\\8g+12f+c=-40\\6g+4f+c=-13\end{cases}$

Use a calculator or otherwise, $g=-\dfrac{15}{2}$, $f=-\dfrac{3}{2}$ and $c=38$

Therefore the equation of the circle is $x^2+y^2+2gx+2fy+c=0$

substitute the above values to get $x^2+y^2-15x-3y+38=0$

EXERCISE 6 COORDINATE 1

1. Find the equation of the circle passing through the following points $(0,0)$, $(3,1)$ and $(3,9)$
2. Find the equation of the diameter of the circle $x^2+y^2-11x-7y+30=0$ which when produced pass through the point $(8,-2)$
3. Find the equation of the circle whose center lies on the line $3x-y-7=0$ and passes through the points $(1,1)$ and $(2,-1)$

4. If O is origin and P, Q are the intersections of the circle $x^2+y^2+4x+2y-20=0$ and the straight line $x-7y+20=0$, then show that lines OP and OQ are perpendicular, and find the equation of the circle through point Q, P and O.

1.17. Equation of a tangent and normal to a circle

1.17.1. Tangent equation

A tangent is the line that touches circle at only one point. The slope of this line at a point of contact with the circle is equals to the slope of the circle at the same point.

Suppose the equation of the circle is $x^2+y^2+2gx+2fy+c=0$ then the slope of the circle at any point is the slope of the tangent at that point. To get the slope we differentiate the circle equation.

Derivative $\quad 2x+2y\dfrac{dy}{dx}+2g+2f\dfrac{dy}{dx}=0$

Slope $\quad \dfrac{dy}{dx}=\mathrm{m}=-\dfrac{x+g}{y+f}$

The slope of the tangent is $\dfrac{dy}{dx}=m=-\dfrac{x+g}{y+f}$

Example 25

Find the equation of the tangent to the circle $x^2+y^2-50=0$ at the point $(-7,1)$

Solution

Given $\quad x^2+y^2-50=0$

Gradient $\quad 2x+2y\dfrac{dy}{dx}=0$

$$\frac{dy}{dx}=-\frac{x}{y}$$

At the point $(x, y) = (-7, 1)$

The slope is $m = -\left(-\dfrac{7}{1}\right) = 7$

$y = m(x - x_1) + y_1$ when $(x_1, y_1) = (-7, 1)$

$$y = 7(x + 7) + 1$$

$$7x - y + 50 = 0$$

The equation of the tangent is $7x - y + 50 = 0$

Example 26

Find the equation of the tangent to the circle $4x^2 + 4y^2 - x + 5y - 23 = 0$ at the point $(2, 1)$

Solution

Given circle $4x^2 + 4y^2 - x + 5y - 23 = 0$

Slope $\quad 8x + 8y\dfrac{dy}{dx} - 1 + 5\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{1 - 8x}{5 + 8y}$$

$$m = \frac{dy}{dx} = \frac{1 - 16}{5 + 8} = -\frac{15}{13}$$

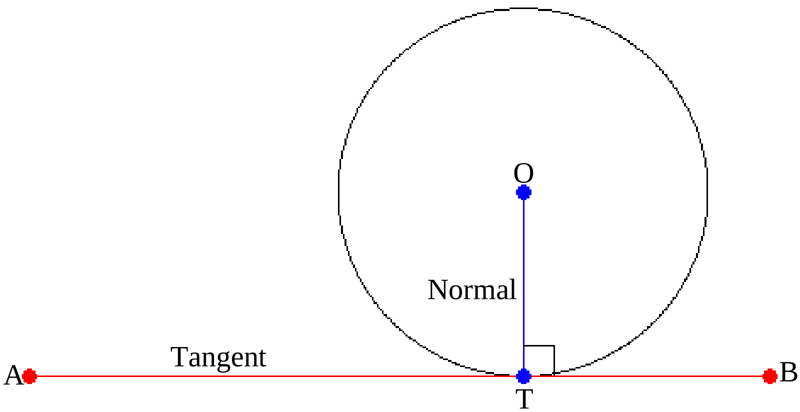
$$y = m(x - x_1) + y_1$$

$$y = -\frac{15}{13}(x - 2) + 1$$

The tangent equation is $15x + 13y - 43 = 0$

1.17.2. Normal equation

Tangent and the Normal are perpendicular, therefore if m_1 is the slope of the tangent, then the slope of the normal is $m_2 = -\frac{1}{m_1}$.



Line *AB* is a tangent to the circle, line *OT* is a normal to the circle, and *T* is a point of contact.

Example 27

Find the equations of the tangent and normal to the circle $x^2+y^2-8x-2y=0$ at the point $(3,5)$

Solution

From the circle $x^2+y^2-8x-2y=0$

The slope of the tangent is $2x+2y\frac{dy}{dx}-8-2\frac{dy}{dx}=0$

$$(2y-2)\frac{dy}{dx}=8-2x$$

The slope of tangent $\frac{dy}{dx}=\frac{8-2x}{2y-2}$

$$\frac{dy}{dx}=\frac{8-2x}{2y-2} \text{ at } (3,5)$$

The slope of the tangent $m_1 = \frac{dy}{dx} = \frac{8-2(3)}{2(5)-2} = \frac{1}{4}$

The equation of the tangent $y = m(x - x_1) + y_1$ at $(3,5)$

$$y = \frac{1}{4}(x-3)+5$$

$$x - 4y + 17 = 0$$

The slope of the normal: $m_2 = -\frac{1}{m_1}$ therefore $m_2 = -4$

The equation is given by $y = -\frac{1}{m_1}(x - x_1) + y_1$

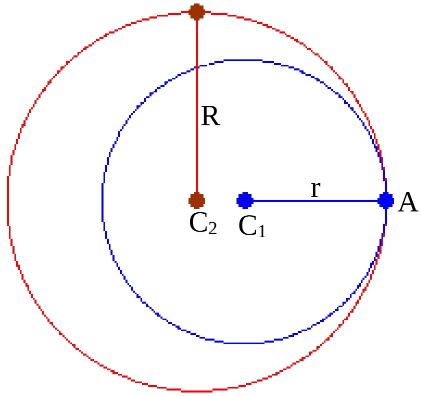
$$y = -4(x-3)+5 \text{ thus } 4x - y - 17 = 0$$

EXERCISE 7 COORDINATE 1

1. Find the equation of the tangent and the normal to the circle $x^2 + y^2 - 26x + 12y + 105 = 0$ at the point $(7,2)$
2. Find the equation of the tangent and the normal to the circle $x^2 + y^2 + 4x + 5y - 20 = 0$ at a point $(-2,3)$
3. Find the condition for which $y = \mathrm{m}x$ touches the circle $x^2 + y^2 + 2gx + 2fy + c = 0$
4. Show that $8x - 11y - 46 = 0$ is the tangent to the circle $x^2 + y^2 + 2x - 7y - 33 = 0$ and find the equation of the diameter through the point of contact.
5. A circle touches the x – axis and cut off a constant length $2a$ from y – axis. Show that the equation to the locus is the curve $x^2 + y^2 = a^2$
6. Show that a general tangent to the circle $x^2 + y^2 = a^2$ may be written as $y = mx \pm a\sqrt{1+m^2}$

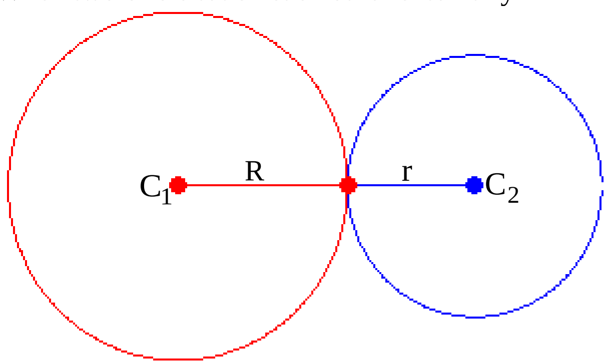
1.18. Intersection of circle

1. When two circles touch the other circle internally



The distance between two centers is $C_1C_2 = R - r$

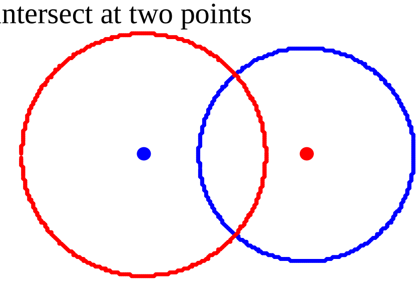
2. When two circle touch each other externally



The distance between the two centers is $\overline{C_1C_2} = R + r$

Remember, the centers are the coordinates, therefore the distance is $\overline{C_1C_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ when $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$

3. When intersect at two points



Example 28

Show that the line $2x+2y-3=0$ touches the circle

$4x^2+4y^2+8x+4y-13=0$

Solution

From $\quad 2x+2y-3=0$

$$y=\frac{3-2x}{2}$$

Substituting

$$4x^2+4y^2+8x+4y-13=0$$

$$4x^2+4\left(\frac{3-2x}{2}\right)^2+8x+4\left(\frac{3-2x}{2}\right)-13=0$$

$$8x^2-8x+2=0$$

If it touches a circle, it means it is a tangent to the circle, the condition $b^2=4ac$ is satisfied.

$b^2=4ac$ then $(-8)^2=4\times8\times2$

Example 29

Show that if $y=mx$ touches the circle $x^2+y^2+2gx+2fy+c=0$ then $c\left(1+m^2\right)=(g+mf)^2$

Solution

Given $\quad x^2+y^2+2gx+2fy+c=0$ and $y=mx$

$$x^2+(mx)^2+2gx+2f(mx)+c=0$$

$$x^2+m^2x^2+2gx+2fmx+c=0$$

$$\left(1+m^2\right)x^2+(2g+2fm)x+c=0$$

Condition $b^2=4ac$

$$(2g+2fm)^2=4\left(1+m^2\right)c$$

$$4(g+fm)^2=4\left(1+m^2\right)c$$

$$(g+fm)^2=\left(1+m^2\right)c$$

Hence shown

1.19. The circle through the points of intersection of other given circles (Family circle)

The equation of the circle through the point of intersection is given by $C_1 + \lambda C_2 = 0$ where C_1 and C_2 are the equations of circles.

Example 30

Find the equation of the circle through the point $(2,-1)$ and through the point of intersection of circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 + 8x - 4y + 6 = 0$

Solution

$$C_1 + \lambda C_2 = 0$$

$$x^2 + y^2 - 2x - 4y - 4 + \lambda\left(x^2 + y^2 + 8x - 4y + 6\right) \text{ at } (2,-1)$$

$$1 + \lambda(29) = 0 \Rightarrow \lambda = -\frac{1}{29}$$

$$x^2 + y^2 - 2x - 4y - 4 - \frac{1}{29}\left(x^2 + y^2 + 8x - 4y + 6\right) = 0$$

$$29x^2 + 29y^2 - 58x - 116y - 116 - x^2 - y^2 - 8x + 4y - 6 = 0$$

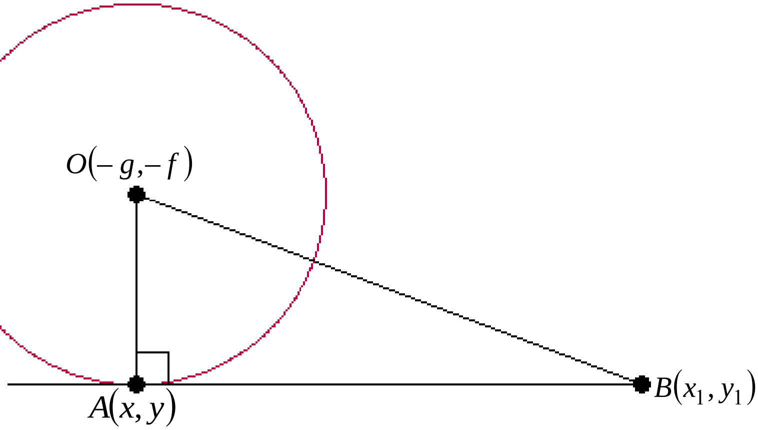
$$14x^2 + 14y^2 - 33y - 56y - 61 = 0$$

The equation required is $14x^2 + 14y^2 - 33y - 56y - 61 = 0$

1.20. Length of the tangent from the external point of the circle

The distance is given by $d = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Proof:



By Pythagoras theorem $\overline{AB}^2+\overline{AO}^2=\overline{OB}^2$

But $\overline{AO}=r$

$\overline{OB}^2=(x_1+g)^2+(y_1+f)^2$

$\overline{OB}^2=x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2$

$r^2=\overline{AO}^2=g^2+f^2-c$

$\overline{AB}^2=\overline{OB}^2-\overline{AO}^2$

$\overline{AB}^2=x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2-g^2-f^2+c$

$\overline{AB}^2=x_1^2+y_1^2+2gx_1+2fy_1+c$

$d=\overline{AB}=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$

Proved.

Example 31

Find the length of the tangent from the point $(3,2)$ to the circle

$x^2+y^2+3x-5y-2=0$

Solution

The length is $d=\sqrt{x_1^2+y_1^2+3x_1-5y_1-2}$ at $(3,2)$

$d=\sqrt{(3)^2+(2)^2+3(3)-5(2)-2}$

$d=\sqrt{10}$ units

1. If $d=0$ the point is on the circle

2. If $d = \sqrt{\text{negative}}$ the point is inside the circle.
3. If $d > r$ the point is outside the circle.

Example 32

Find the distance from the circle $2x^2 + 2y^2 - 3x + 5y - 84 = 0$ to the point

(a) $(3, 5)$ (b) $(2, -2)$

Solution

(a) $d = \sqrt{2x_1^2 + 2y_1^2 - 3x_1 + 5y_1 - 84}$ at $(3, 5)$

$d = \sqrt{2(3)^2 + 2(5)^2 - 3(3) + 5(5) - 84}$

$d = \sqrt{84 - 84}$ then $d = 0$

The distance is zero; therefore, the point is on the circle.

(b) $d = \sqrt{2x_1^2 + 2y_1^2 - 3x_1 + 5y_1 - 84}$ at $(2, -2)$

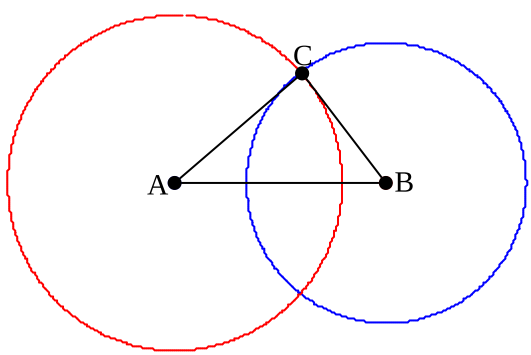
$d = \sqrt{2(2)^2 + 2(-2)^2 - 3(2) + 5(-2) - 84}$

$d = \sqrt{-84}$

The distance is complex; therefore, the point is inside the circle.

1.21. Orthogonal circles

These are the circles whose tangents forms the right angle at their point of contact.



$\angle ACB = 90°$

Where A and B are centers of the circles. By Pythagoras theorem,

$\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$

Let the equation of the circle with center A be

$x^2 + y^2 + 2gx + 2fy + c = 0$ and that of center B be

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ therefore, $A(-g, -f)$ and $B(-g_1, -f_1)$

$\overline{AB}^2 = (g - g_1)^2 + (f - f_1)^2$

$\overline{AB}^2 = g^2 - 2gg_1 + g_1^2 + f^2 - 2ff_1 + f_1^2$

$\overline{AC}^2 = g^2 + f^2 - c$

$\overline{BC}^2 = g_1^2 + f_1^2 - c_1$

$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$

$g^2 - 2gg_1 + g_1^2 + f^2 - 2ff_1 + f_1^2 = g^2 + f^2 - c + g_1^2 + f_1^2 - c_1$

$-2gg_1 - 2ff_1 + c + c_1 = 0$

The condition for orthogonal circle is $2gg_1 + 2ff_1 = c + c_1$

Example 33

Show that the circles $x^2 + y^2 - 6y + 8 = 0$ and

$x^2 + y^2 - 4x + 2y - 14 = 0$

Solution

The distance from the centers of the two circles is equal to the sum of the square of radii.

From the circle: $x^2 + y^2 - 6y + 8 = 0$

$$x^2 + (y-3)^2 + 8 - 9 = 0$$

$$x^2 + (y-3)^2 = 1$$

Radius of first circle is $r_1 = 1$ and the center is $C_1(0, 3)$

From the circle: $x^2 + y^2 - 4x + 2y - 14 = 0$

$$(x-2)^2 + (y+1)^2 - 19 = 0$$

$$(x-2)^2+(y+1)^2=19$$

The radius of the second circle is $r_2=\sqrt{19}$

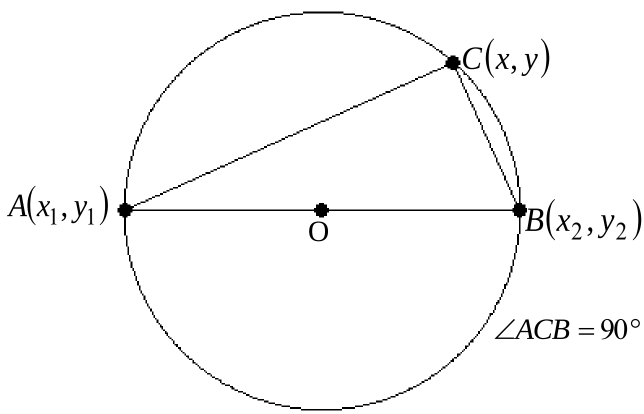
It is center is $C_2(2,-1)$

$$\overline{C_1C_2}^2=r_1^2+r_2^2$$
$$\overline{C_1C_2}^2=(0-2)^2+(-1-3)^2$$
$$\overline{C_1C_2}^2=20$$
$$r_1^2+r_2^2=\left(\sqrt{1}\right)^2+\left(\sqrt{19}\right)^2$$
$$r_1^2+r_2^2=20$$

Therefore, $\overline{C_1C_2}^2=r_1^2+r_2^2$ the circles are orthogonal.

1.22. Equation of circle given the ending points of the diameter

Consider the figure below



Line AB is a diameter of the circle with the end points $A(x_1,y_1)$ and $B(x_2,y_2)$. Assume any point $C(x,y)$ on the circumference. Angle ACB is the right angle, therefore lines AC and line BC are perpendicular to each other.

Slope of BC $m_1=\dfrac{y_2-y}{x_2-x}$ and the slope of AC $m_2=\dfrac{y_1-y}{x_1-x}$

Then, $m_1 m_2 = -1$

$$\left(\frac{y_2 - y}{x_2 - x}\right)\left(\frac{y_1 - y}{x_1 - x}\right) = -1$$

$$(x_2 - x)(x_1 - x) + (y_2 - y)(y_1 - y) = 0$$

This is the equation of the circle given the end points of the diameter.

Example 34

Find the equation of the circle with the endpoints of the diameter $A(-6, 3)$ and $B(-3, -4)$

Solution

Equation of the circle given endpoints of the diameter is given by

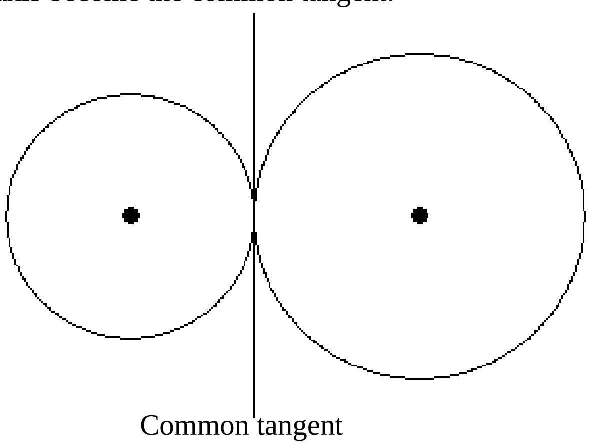
$$(x_2 - x)(x_1 - x) + (y_2 - y)(y_1 - y) = 0$$

Therefore, $(x + 3)(x + 6) + (y + 4)(y - 3) = 0$

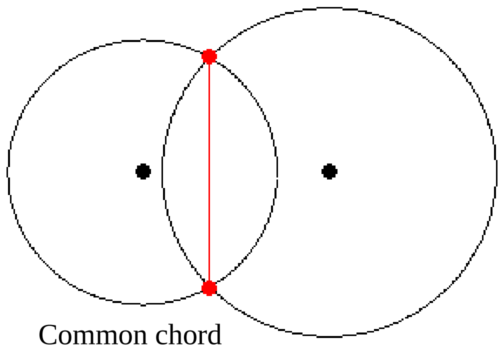
$$x^2 + y^2 + 9x + y + 6 = 0$$

1.23. Common chord and Radical axis of circles (Power line)

The radical axis of the two circle is the locus of a point, which moves so that the length of the tangent drawn form it to the two circles are equal. The radical axis is perpendicular to the radii of the circles (the line joining the centres of the circles). When circles are tangents to each other the radical axis become the common tangent.



When circle intersect



Common chord

To get the radical axis find the difference between the circle equations

Assume the two circles $\begin{cases} x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \\ x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \end{cases}$

Then the radical axis is given by $-\begin{cases} x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \\ x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \end{cases}$

$2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

This is the equation of the straight line.

Example 35

Find the equation of the radical axis of the following circles

$x^2 + y^2 + 3x - 4y - 12 = 0$ and $x^2 + y^2 - 2x - 5y + 7 = 0$

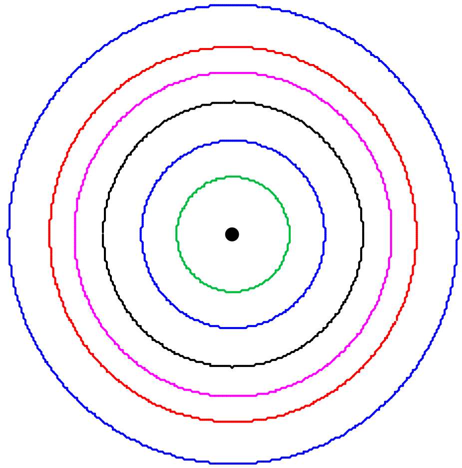
Solution

Radical axis $= -\begin{cases} x^2 + y^2 + 3x - 4y - 12 = 0 \\ x^2 + y^2 - 2x - 5y + 7 = 0 \end{cases}$

The equation of radical axis is $5x + y - 19 = 0$

1.24. Concentric circle

The concentric circles are circles with the common center. The figure below portrait the concept of family circles. What is important here is all the circles will have the same equation except the value of *"c"* which varies from circle to circle. The circles



$x^2+y^2+2gx+2fy+c_1=0$, $x^2+y^2+2gx+2fy+c_2=0$ and $x^2+y^2+2gx+2fy+c_3=0$ are family circles.

All these circles has their center at point (4, 6). This kind of circles are called family circles

1.25. Equations of concentric circle

Consider the following circles

$$3x^2+3y^2+10x-12y+3=0,$$

$$3x^2+3y^2+10x-12y+A=0,$$

$$3x^2+3y^2+10x-12y+B=0,$$

$$3x^2+3y^2+10x-12y+C=0,$$

$$3x^2+3y^2+10x-12y+D=0$$

All these circles are centred at $\left(-\frac{5}{3},2\right)$, but radii are different, only if

$$3\neq A\neq B\neq C\neq D$$

Example 36

Find the equation of the circle which is concentric with the circle $x^2+y^2+14x-10y-5=0$ whose radius is $2\sqrt{3}$ units.

Solution

$$x^2+y^2+14x-10y-5=0$$

$$(x+7)^2-49+(y-5)^2-25-5=0$$

$$(x+7)^2+(y-5)^2-74-5=0$$

$$(x+7)^2+(y-5)^2=79$$

The centre of these circle is $C(-7,5)$

The equation of the other circle is given by

$$(x+7)^2+(y-5)^2=\left(2\sqrt{3}\right)^2$$

$$x^2+14x+49+y^2-10y+25=12$$

$$x^2+y^2+14x-10y+62=0$$

1.26. Triangles properties

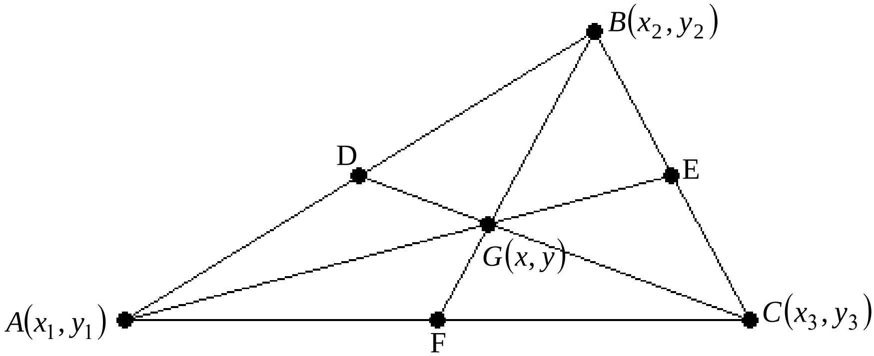
1.26.1. Centroid of the triangle

The centroid of the triangle is the point where the median of the triangle meets, it is sometimes called center of gravity of the triangle. Medians of the triangle are the lines, which bisect the sides of the triangle and meet at a point inside a triangle.

Properties of the centroid

1. It is formed by the concurrency of the median of the triangle
2. It is located inside the triangle
3. It divided median into ratio 2:1

Consider the figure below



By using the mid-point formula, we can determine the coordinates of points D, E and F as follows:-

The coordinate of point D is $\left(\dfrac{x_2 + x_1}{2}, \dfrac{y_2 + y_1}{2}\right)$

The coordinate of point E is $\left(\dfrac{x_2 + x_3}{2}, \dfrac{y_2 + y_3}{2}\right)$

The coordinate of point F is $\left(\dfrac{x_1 + x_3}{2}, \dfrac{y_1 + y_3}{2}\right)$

To get the coordinate of G, use the internal division of the line formula, remember, $\overline{AG} : \overline{GE} = 2:1$ for centroid of the triangle

$$G(x,y) = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right)$$

Where $m_1 = 2$ and $m_2 = 1$ between the points $A(x_1, y_1)$ and $\left(\dfrac{x_2 + x_3}{2}, \dfrac{y_2 + y_3}{2}\right)$

$$G(x, y) = \left(\frac{2\left(\frac{x_2 + x_3}{2}\right) + x_1}{2+1}, \frac{2\left(\frac{y_2 + y_3}{2}\right) + y_1}{2+1}\right)$$

$$G(x, y) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Therefore, the centroid of the triangle is given by

$$G(x,y)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$$

Example 37

Find the centroid of the triangle with the following vertices $P(-2,\ 4)$, $Q(-4,\ 8)$ and $R(3,\ 9)$

Solution

The centroid is given by $G(x,y)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$

$$G(x,y)=\left(\frac{-2+-4+3}{3},\frac{4+8+9}{3}\right)$$

$$G(x,y)=\left(-\frac{3}{3},\frac{21}{3}\right)=(-1,7)$$

The centroid of the triangle is $(-1,7)$

EXERCISE 8 – COORDINATE 1

1. Find the centroid of the triangle with the coordinates $A(2,-1)$, $B(3,2)$ and $C(9,3)$ at the vertices
2. If a triangle has the centroid of $(3,-4)$ find the missing vertex coordinate if the other vertices has $(10,4)$ and $(1,7)$
3. Given $d=\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$, if $x=a$, $y=b$ and $d=2\sqrt{c}$, show that $a^2+b^2=c$
4. If $y=mx+c$ is the tangent of the circle $x^2+y^2=r^2$, show that $c=\pm r\sqrt{m^2+1}$
5. Find the point of intersection of circles $x^2+y^2+2x-6y+2=0$ and $x^2+y^2+8x-6y+20=0$.
6. Find the equation of the circle which endpoints of the diameter $(-1,2)$ and $(-1,5)$

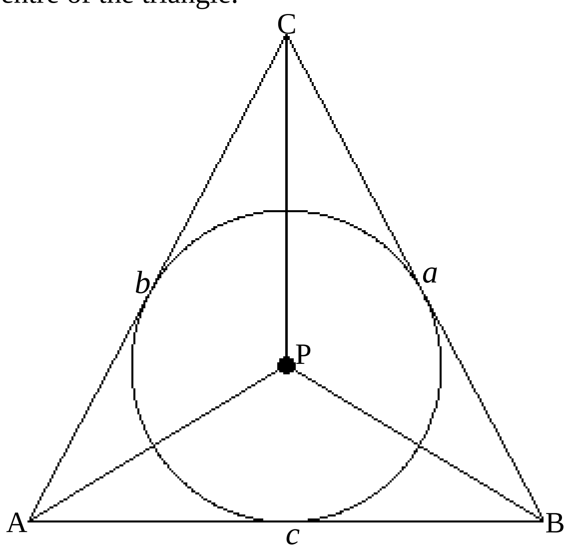
7. Find the circle with radius $3\sqrt{5}$ units and concentric to the circle $9x^2+9y^2-36x-6y-107=0$

8. The circles $x^2+y^2+8x+ay+43=0$ and $2x^2+2y^2+16x-24y+b=0$ are concentric, find the values of *a* and *b*, hence find their centre.

1.26.2. Incentre of the triangle

Incentre of the triangle is a point where three bisectors of the triangle meet.

To get the incentre inscribe the triangle in a circle and center of the circle will be the incentre of the triangle.



Line *PC* bisect angle *QPR*, line *CR* bisect angle *QRP* and line *QC* bisect angle *PQR*.

The incentre is given by the formula

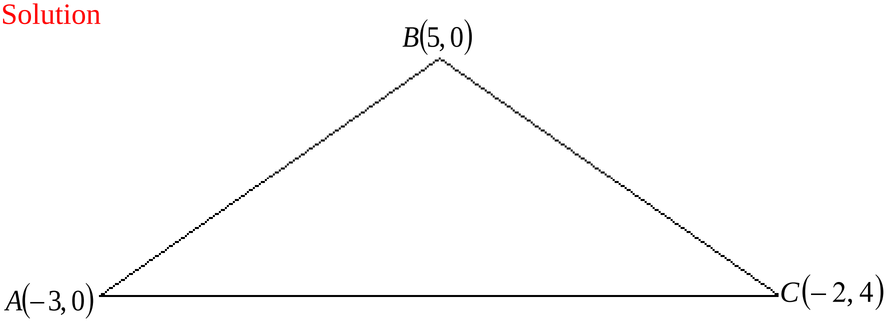
$$P(x,y)=\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

Where *a*, *b*, and *c* are the length of the opposite sides of the vertex.

Example 38

Find incentre of the triangle whose vertices are $A(-3,0)$, $B(5,0)$ and $C(-2,4)$

Solution



$$c=\overline{AB}=\sqrt{(-3-5)^2}=8$$

$$b=\overline{AC}=\sqrt{(-3+2)^2+(0-4)^2}=\sqrt{17}\approx 4$$

$$a=\overline{BC}=\sqrt{(5+2)^2+(0-4)^2}=\sqrt{65}\approx 8$$

Incentre $C(-2,4)$

$$P(x,y)=\left(\frac{8\times-3+4\times-2+8\times5}{8+4+8},\frac{8\times0+4\times4+8\times0}{8+4+8}\right)$$

The incentre is $P(x,y)=\left(0,\frac{4}{5}\right)$

MISCELLANEOUS EXERCISE – COORDINATE 1

1. The vertex B of a square $ABCD$ has the coordinate $(-1,-3)$. The diagonal AC lies along the line $2y = x + 5$

(a) Show that the coordinates of D are $(-5, 5)$ and find the coordinates of A and C

(b) Show that the equation of the circle touching all the four points of the square passes through the origin

(c) Calculate the area of the square which lies in the first quadrant

(d) Find the equation of the circle which is concentric to the circle in square above and find its radius.

2. The coordinates $A(4, 4)$, $B(-4, 0)$ and $C(6, 0)$ are the coordinates of the vertices of a triangle.

(a) Find the coordinate of the point where the internal bisector of the angle BAC meets the line $y = 0$

(b) Find the equation of the circle inscribing this triangle

(c) Find the equation of the circle which touches line AC at point C and pass-through point B

3. (a) Find the equation of the circle which passes through the point $(-4, 1)$ and the center is at the origin.

(a) Line $3x + y = 10$ is a tangent to the circle whose center is at the origin, find the equation of this circle, hence show that the shortest distance from the center to the line $3x + y = 10$ is the radius of the circle.

(b) Find the equation of the circle with line $3x + y = 10$ as a tangent and the center at $(5, 7)$.

4. (a) Find the equation of the circle which has the line segment $(3,-4)$ and $(-3, 4)$ as the dimeter.

(b) Show that if the radius of the circle $x^2 + y^2 - 6x + 4ky + 20 = 0$ is 5 units, then $k = 3$ or $k = -3$ hence find the possible centers of the circle.

5. Determine if the following points $A(-3,-2)$, $B(5,-1)$, $C(-2, 1)$ and $D(-2, 3)$ are inside, on or outside the circle $x^2 + y^2 - 2x + 8y - 8 = 0$

6. Find the center and radius of the following circles

(a) $x^2 + y^2 - 2x + 4y - 4 = 0$

(b) $2x^2+2y^2-2x-6y-13=0$

(c) $9x^2+9y^2-6x+54y+46=0$

(d) $x^2+y^2+6x-7=0$

(e) $x^2+y^2-10x+2y+6=0$

(f) $(x-3)(x+3)+(y+2)(y+6)=0$

(g) $(x-2)(x+4)+(y-1)(y-5)=0$

7. Prove that the points $A(-2,-1)$, $B(4,3)$, $C(6,0)$ and $D(0,-4)$ form the vertices of a rectangle. The straight line $x=3$ meets the sides *AB* and *DC* in *P* and *Q* respectively. Calculate the area of the trapezium *PBCQ*.

8. (a) Show that the circle $x^2+y^2-2\lambda x+(\lambda-2a)^2=0$ touches the parabola $y^2=4ax$, $a>0$ at two real points, for all values of λ greater that $2a$.

(b) (i) Find the value of λ greater than $2a$ for which the circle will pass through the focus of the parabola.

(ii) In this case, if *A* is the center of the circle and the parabola, show that a second circle having center *P* and radius *PA* will touch the directrix of the parabola.

9. A curve has parametric equations: $x=t^2+1$ and $y=t^3$, show that $(5,-8)$ lies on the curve, and find the equation of the tangent at this point.

10. (a) The curve $y^2=12x$ intersects the line $3y=4x+6$ at two points. Find the distance between the two points and the equation of the circle which passes through the point of intersections and the end points of the diameter.

(b) In a circle of center *O* and radius *r*, *OP* and *OQ* are two radii enclosing a small angle. Tangents to the circle at P and *Q* intersects at *R*. If the length of the arc *PQ* is *x*, show that the area enclosed by *PR*, *QR* and an arc *PQ* is approximately $x^3/24r$.

11. (a) An equilateral triangle *ABC* has sides of length 6 cm. The three altitudes of the triangle meet at *N*. Show that $AN=2\sqrt{3}$ cm

(b) This triangle is the base of a pyramid whose apex *V* lies on the line through *N* perpendicular to the plane *ABC*. Given that $VN=2$ cm, show that $\angle VAN=30°$

(c) The perpendicular from A to the edge VC meets CV produced at R. Prove that $AR = \frac{3}{2}\sqrt{7}$ cm and find the value of $\cos(\angle ARB)$

12. The straight-line l has equation $2y - x + 7 = 0$. The straight-line l' passes through the point $P(-1, 6)$ and is perpendicular to l.

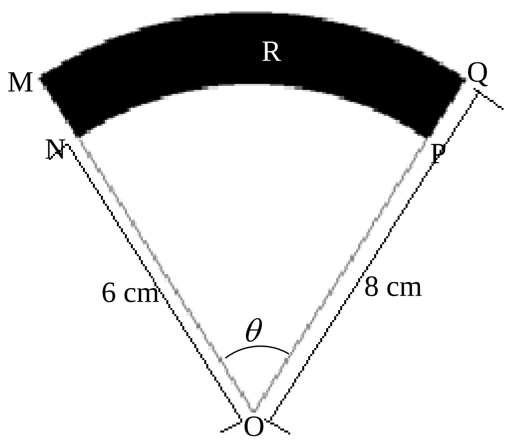
(a) Find the equation of l', giving your answer in the form $ax + by + c = 0$

(b) Find the coordinates of the point of intersection of l and l'.

(c) Show that the perpendicular distance from P to l is $4\sqrt{5}$

(d) It is given that the points $Q(-7, -7)$ and $R(9, 1)$ lie on l. Find the area of the triangle PQR.

13. The diagram shows a sector OMQ of the circle with center O and radius 8 cm. An arc of the circle with centre O and radius 6 cm joins the point N on OM and P on OQ. Angle NOP is θ, where $0° < \theta < 180°$, and the area of the shaded region R between the two arcs is $A\text{cm}^2$. Express A in terms of θ (θ is in degree).



It is given that θ is increasing at a rate of 0.1rad/sec

(a) Find the rate of increase of A with respect to time

(b) Find the rate of increase of the perimeter of the region R with respect to time

(c) The length of the straight-line NQ is L cm. Show that $L^2 = 100 - 96\cos\theta$

(d) Find the rate of increase of L with respect to time when $\theta = \frac{1}{3}\pi$

14. The curve C has polar equation $r\theta = 1$, for $0 < \theta \le 2\pi$

(a) Use the fact that $\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1$ to show that the line with Cartesian equation $y=1$ is the asymptote to C

(b) Sketch C

(c) The point P and Q on C correspond to $\theta=\frac{1}{6}\pi$ and $\theta=\frac{1}{3}\pi$ respectively

(i) Find the area of the sector OPQ, where O is the origin

(ii) Show that the length of the arc PQ is $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}\frac{\sqrt{1+\theta^2}}{\theta^2}\,d\theta$

15. Sketch one loop of the curve whose polar equation is $r=a\sin 2\theta$ on the first quadrant, where a is a positive constant. Find the area of the loop, giving your answer in terms of a and π

16. Find the coordinates of the point common to the curves $y=x^2-1$ and $y=\left(x^2-1\right)^3$

17. Without drawing find the gradients of the two lines through the origin which makes an angle of $45°$ with the line $y=2x$

18. A square is to be constructed with one vertex at the origin *O*, and with one diagonal lying along the line $y=2x$. If one of its sides (produced if necessary) is to pass through the point $(3,5)$ find, for each of the two possible squares, the equation of the second diagonal.

19. The base *AB* of a triangle *ABC* is fixed, and *K* is fixed point on *AB*. The vertex *C* of the triangle moves so that the perpendicular distances of *K* from *CA* and *CB* are always equal in length. Prove that, in general, the locus of *C* is a circle through *K*.

20. A small radio transmitter broadcasts in a 50-mile radius. If you drive along a straight line from a city 60 miles north of the transmitter to a second city 70 miles east of the transmitter, during how much of the drive will you pick up a signal from the transmitter?

21. List areas where you think coordinates geometry is used in our real-life situation.

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Advanced Mathematics

Coordinate Geometry 1

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