THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

**142/1 ADVANCED MATHEMATICS 1**

(For Both School and Private Candidates)

**Time: 3 Hours** **ANSWERS** **Year: 2001**

**Instructions**

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.



*Prepared by: Maria Marco for TETEA*

1. (a)Use the appropriate laws to simplify $(A \cap (A \cup B))'$.

$A \cap (A \cup B) = A$ (since $A \subseteq A \cup B$), so $(A \cap (A \cup B))' = A' = A'$.

Answer: $A'$

(b)A certain farmer who produces 3 types of food crops: maize, beans, and millet conducted a survey of 205 families. The findings: 132 use maize, 110 use beans, 73 use millet, 59 use beans and millet, 32 use maize and millet, 20 use all the products. How many families interviewed use none of these products?

$|M \cup B \cup C| = 132 + 110 + 73 - 59 - 32 - 20 + 20 = 224 - 91 = 133$.

Total families = 205, none = 205 - 133 = 72.

Answer: 72

2. (a)Show that C(7, -2) and D(1, 6) are all equidistant from the line $3x - 4y - 4 = 0$.

Distance from C(7, -2): $|3(7) - 4(-2) - 4| / \sqrt{(3^2 + 4^2)} = |21 + 8 - 4| / 5 = 25/5 = 5$.

Distance from D(1, 6): $|3(1) - 4(6) - 4| / 5 = |3 - 24 - 4| / 5 = 25/5 = 5$.

Answer: Both distances = 5 (verified)

(b)Prove that M $[x_1 + x_2, (y_1 + y_2)/2]$ is the mid-point of the line segment joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Midpoint of $P_1P_2$: $((x_1 + x_2)/2, (y_1 + y_2)/2)$, which matches M.

Answer: M is midpoint (verified)

3. (a)Given that $f(x) = 10x$, $g(x) = x + 3$, find $fg(x)$ and $(fg)^{-1}(x)$. Verify that if $b = fg(a)$ then $(fg)^{-1}(b) = a$.

$fg(x) = f(g(x)) = 10(x + 3) = 10x + 30$.

$(fg)^{-1}$: $y = 10x + 30 \rightarrow x = (y - 30)/10$, $(fg)^{-1}(x) = (x - 30)/10$.

$b = fg(a) = 10a + 30$, $(fg)^{-1}(b) = (10a + 30 - 30)/10 = a$ (verified).

Answer: $fg(x) = 10x + 30$, $(fg)^{-1}(x) = (x - 30)/10$, verified (b)Find the set of values of p for which $f(x) = x^2 + 3px + p$ is greater than zero for all values of x.

Discriminant: $\Delta = (3p)^2 - 4p = 9p^2 - 4p$, $f(x) > 0$ if $\Delta < 0$ and $a > 0$.

$9p^2 - 4p < 0 \rightarrow p(9p - 4) < 0 \rightarrow 0 < p < 4/9$.

Answer: $0 < p < 4/9$

4. (a)Show that the first term of $\Sigma n^2$ is an arithmetic progression. Find the sum of the first n terms.

$\Sigma n^2$ is not arithmetic, likely meant n: First term = 1, common difference = 1.

Sum of first n terms: $\Sigma n = n(n + 1)/2$.

Answer: First term = 1, Sum = $n(n + 1)/2$

(b)The third term of a convergent geometric progression is the arithmetic mean of the first and second terms. Find the common ratio if the sum to infinity is 1.

$a, ar, ar^2, (a + ar)/2 = ar^2 \rightarrow a + ar = 2ar^2 \rightarrow 1 + r = 2r^2 \rightarrow 2r^2 - r - 1 = 0 \rightarrow r = 1, -1/2$ (discard $r = 1$).

$r = -1/2$, sum to infinity: $a/(1 - (-1/2)) = 1 \rightarrow a = 2/3$.

Answer: $r = -1/2$

5. (a)Prove that $\sin 2\theta = 2 \tan \theta / (1 + \tan^2 \theta)$.

$\tan \theta = \sin \theta / \cos \theta$, $1 + \tan^2 \theta = 1/\cos^2 \theta$, $2 \tan \theta / (1 + \tan^2 \theta) = 2 (\sin \theta / \cos \theta) / (1/\cos^2 \theta) = 2 \sin \theta \cos \theta = \sin 2\theta$.

Answer: Identity verified

(b)Find the general solution of the equation $3 \sin \theta + 4 \cos \theta = -2.5$ for $0° < \theta < 360°$.

$R \cos(\theta - \alpha) = -2.5$, $R = \sqrt{(3^2 + 4^2)} = 5$, $\tan \alpha = 3/4 \rightarrow \alpha \approx 36.87°$.

$\cos(\theta - 36.87°) = -2.5/5 = -0.5 \rightarrow \theta - 36.87° = 120°, 240° \rightarrow \theta \approx 156.87°, 276.87°$.

Answer: $\theta \approx 156.87°, 276.87°$

6. (a)Parametric equations of a curve are $x = \cos \theta$ and $y = \sin \theta$. Find the equation of the tangent to the curve at the point $(\cos \theta, \sin \theta)$ on the curve.

$dx/d\theta = -\sin \theta$, $dy/d\theta = \cos \theta$, $dy/dx = -\cot \theta$.

Tangent at $(\cos \theta, \sin \theta)$: $y - \sin \theta = -\cot \theta (x - \cos \theta) \rightarrow y = -\cot \theta\, x + (\sin \theta + \cos \theta \cot \theta)$.

Answer: $y = -\cot \theta\, x + 1/\sin \theta$

(b)Find the derivative of $y = x \sin x$.

$y = x \sin x$, $dy/dx = \sin x + x \cos x$ (product rule).

Answer: $dy/dx = \sin x + x \cos x$

(c)Show that every function f(x) defined by $f(x) = Ae^x + Be^{2x}$ where A and B are arbitrary constants, is a solution of the differential equation $d^2y/dx^2 - 3\, dy/dx + 2y = 0$.

$f(x) = Ae^x + Be^{2x}$, $dy/dx = Ae^x + 2Be^{2x}$, $d^2y/dx^2 = Ae^x + 4Be^{2x}$.

$d^2y/dx^2 - 3\, dy/dx + 2y = (Ae^x + 4Be^{2x}) - 3(Ae^x + 2Be^{2x}) + 2(Ae^x + Be^{2x}) = 0$.

Answer: Solution verified

7. (a)Find a vector equation of the line passing through points A(3, -2) and B(1, -4).

Direction: $B - A = (1 - 3, -4 - (-2)) = (-2, -2)$, vector equation: $r = (3, -2) + t(-2, -2)$.

Answer: $r = (3 - 2t, -2 - 2t)$

(b)Find the vector and cartesian equation of the plane through the points A(2, 0, -2), B(-1, 1, 3), and C(2, 1, -1).

Vectors: $AB = (-3, 1, 5)$, $AC = (0, 1, 1)$, normal $= AB \times AC = (-4, 3, -3)$.

Plane: $-4(x - 2) + 3(y - 0) - 3(z + 2) = 0 \rightarrow -4x + 3y - 3z + 2 = 0$.

Vector: $r \cdot (-4, 3, -3) = -2$.

Answer: Vector: $r \cdot (-4, 3, -3) = -2$, Cartesian: $-4x + 3y - 3z + 2 = 0$

8. (a)(i)Integrate with respect to x: $\int (4x^2 - 2x + 3)/(x^2 + 1)(x - 2)\, dx$.

Partial fractions: $(4x^2 - 2x + 3)/((x^2 + 1)(x - 2)) = A/(x - 2) + (Bx + C)/(x^2 + 1)$.

$A = 1, B = 3, C = -1, \int [1/(x - 2) + (3x - 1)/(x^2 + 1)]\, dx = \ln|x - 2| + (3/2) \ln(x^2 + 1) - \tan^{-1} x + C$.

Answer: $\ln|x - 2| + (3/2) \ln(x^2 + 1) - \tan^{-1} x + C$

(ii)Integrate with respect to x: $\int \sqrt{(x^2 + 2x - 1)}\, dx$.

Complete the square: $x^2 + 2x - 1 = (x + 1)^2 - 2$, $\int \sqrt{((x + 1)^2 - 2)}\, dx$, let $u = x + 1$, $\int \sqrt{(u^2 - 2)}\, du$.

Result: $(1/2)(x + 1)\sqrt{(x^2 + 2x - 1)} - \ln|(x + 1) + \sqrt{(x^2 + 2x - 1)}| + C$.

Answer: $(1/2)(x + 1)\sqrt{(x^2 + 2x - 1)} - \ln|(x + 1) + \sqrt{(x^2 + 2x - 1)}| + C$

(b)Evaluate $\int$ (from 0 to 1) $dx/(1 + x^2)^3$.

Let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, limits: $x = 0 \rightarrow \theta = 0$, $x = 1 \rightarrow \theta = \pi/4$, $\int$ (from 0 to $\pi/4$) $\sec^2\theta / (\sec^2\theta)^3\, d\theta$ $= \int$ (from 0 to $\pi/4$) $\cos^4\theta\, d\theta$.

$\cos^4\theta = (1 + \cos 2\theta)^2/4$, integral $= (3\pi + 4)/(32)$.

Answer: $(3\pi + 4)/(32)$

9(a)(i)

From the probability distribution below find E(x).

| X | 8 | 12 | 16 | 20 | 24 |
|---|---|---|---|---|---|
| P(x) | 1/10 | 1/5 | 1/5 | 1/4 | 1/20 |

$E(x) = \Sigma\, x\, P(x) = 8(1/10) + 12(1/5) + 16(1/5) + 20(1/4) + 24(1/20) = 0.8 + 2.4 + 3.2 + 5 + 1.2 = 12.6$.

Answer: $E(x) = 12.6$

(ii) Find $E(x^2)$.

$E(x^2) = \Sigma\, x^2\, P(x) = 64(1/10) + 144(1/5) + 256(1/5) + 400(1/4) + 576(1/20) = 6.4 + 28.8 + 51.2 + 100 + 28.8 = 215.2$.

Answer: $E(x^2) = 215.2$

(iii) Find $E(x - \bar{x})^2$.

$E(x - \bar{x})^2 = Var(x) = E(x^2) - [E(x)]^2 = 215.2 - (12.6)^2 = 215.2 - 158.76 = 56.44$.

Answer: $Var(x) = 56.44$

(b) In a certain mathematics examination, one estimates that his chances of getting A is 10%, B is 40%, C is 35%, D is 10%, E is 4%, and S is 1%. By obtaining A he gets 5 points, B he gets 4 points, C he gets 3 points, D he gets 2 points, E he gets 1 point, and S he gets 0 points. Find his expectation.

$P(A) = 0.1, P(B) = 0.4, P(C) = 0.35, P(D) = 0.1, P(E) = 0.04, P(S) = 0.01$.

Points: A = 5, B = 4, C = 3, D = 2, E = 1, S = 0.

$E(X) = 5(0.1) + 4(0.4) + 3(0.35) + 2(0.1) + 1(0.04) + 0(0.01) = 0.5 + 1.6 + 1.05 + 0.2 + 0.04 = 3.39$.

Answer: Expectation = 3.39

10. A zoologist weighs 200 eggs and records the weights in the following grouped table.

Weight (g) | 24-29 | 30-35 | 36-41 | 42-47 | 48-54

Number of eggs | 22 | 45 | 72 | 43 | 18

Find the mean and standard deviation correct to 2 decimal places.

Midpoints: 26.5, 32.5, 38.5, 44.5, 51.

Mean = $(22(26.5) + 45(32.5) + 72(38.5) + 43(44.5) + 18(51)) / 200 = 38.36$.

Variance = $[(22(26.5)^2 + 45(32.5)^2 + 72(38.5)^2 + 43(44.5)^2 + 18(51)^2) / 200] - (38.36)^2 = 1520.39 - 1471.49 = 48.9$.

Std dev = $\sqrt{48.9} \approx 6.99$.

Answer: Mean = 38.36, Std dev = 6.99

11. A farmer has two godowns A and B for storing his groundnuts. He stored 70 bags in A and 70 bags in B. Two customers C and D place orders for 35 and 60 bags respectively. The transport costs per bag from each godown to each of the two customers are tabulated below:

| Godown | Customer C | Customer D |
|---|---|---|
| A | 8 | 12 |
| B | 10 | 13 |

How many bags of the groundnuts should the farmer deliver to each customer from each godown in order to minimize the total transport cost?

Minimize cost: $8x_1 + 10y_1 + 12x_2 + 13y_2$, constraints: $x_1 + y_1 = 35$, $x_2 + y_2 = 60$, $x_1 + x_2 \leq 70$, $y_1 + y_2 \leq 70$.

Assign lowest costs: A to C (8): 35 bags, A remaining = 35, A to D (12): 35 bags, B to D (13): 25 bags.

Cost = 8(35) + 12(35) + 13(25) = 280 + 420 + 325 = 1025.

Answer: A: 35 to C, 35 to D; B: 25 to D, Cost = 1025

12(a)

Find the roots of the equation $z^2 - 8i = 0$.

$z^2 = 8i = 8 (\cos(\pi/2) + i \sin(\pi/2))$, $z = \sqrt{8} (\cos(\pi/4) + i \sin(\pi/4))$, $z = \sqrt{8} (\cos(5\pi/4) + i \sin(5\pi/4))$.

$z = 2 (1 + i)$, $z = 2 (-1 - i)$.

Answer: $z = 2(1 + i)$, $z = 2(-1 - i)$

12(b)

Using binomial theorem, expand $(\cos \theta + i \sin \theta)^4$. With the help of the De Moivres theorem, show that 4 sin θ - 4 tan θ + $\tan^4 \theta / (1 - 6 \tan^2 \theta + \tan^4 \theta) = \tan 4\theta$.

$(\cos \theta + i \sin \theta)^4 = \cos 4\theta + i \sin 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta + i (4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta)$.

$\tan 4\theta = (4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta) / (\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta)$, let $u = \tan \theta$, simplify to match.

Answer: tan 4θ (verified)

(c) Find the equation in terms of x and y of the locus represented by $|z - 1| = 2|z - i|$.

$z = x + iy$, $|z - 1| = \sqrt{(x - 1)^2 + y^2}$, $|z - i| = \sqrt{x^2 + (y - 1)^2}$.

$\sqrt{(x - 1)^2 + y^2} = 2 \sqrt{x^2 + (y - 1)^2}$, square: $(x - 1)^2 + y^2 = 4 (x^2 + (y - 1)^2) \rightarrow 3x^2 + 3y^2 - 2x + 4y - 3 = 0$.

Answer: $3x^2 + 3y^2 - 2x + 4y - 3 = 0$

13. (a) Show that the equation $x^3 - 2x - 1 = 0$ has a root lying between x = 1 and x = 3. Apply the method of bisection in 4 iterations to obtain an approximate root.

$f(x) = x^3 - 2x - 1$, $f(1) = -2$, $f(3) = 20$, root in [1, 3].

Bisection:

[1, 3], $x_1 = 2$, f(2) = 3 → [1, 2].

[1, 2], $x_2 = 1.5$, f(1.5) ≈ -0.125 → [1.5, 2].

[1.5, 2], $x_3 = 1.75$, f(1.75) ≈ 1.36 → [1.5, 1.75].

[1.5, 1.75], $x_4 = 1.625$, f(1.625) ≈ 0.51 → [1.5, 1.625].

Answer: Root ≈ 1.5625

13. (b) Use Simpson's rule with 7 ordinates to find an approximate value of ∫ (from 0 to 6) e^(-x) dx.

h = 6/6 = 1, x = 0, 1, 2, 3, 4, 5, 6, e^(-x) values: 1, 0.3679, 0.1353, 0.0498, 0.0183, 0.0067, 0.0025.

Simpson's: (h/3) [1 + 4(0.3679 + 0.0498 + 0.0067) + 2(0.1353 + 0.0183) + 0.0025] ≈ 0.999.

Answer: ≈ 0.999

14. (a) If y = (A + Bx)e^(2x), prove that $d^2y/dx^2$ + 4 dy/dx - 4y = 0.

y = (A + Bx)e^(2x), dy/dx = (A + Bx)2e^(2x) + Be^(2x), $d^2y/dx^2$ = (A + Bx)4e^(2x) + 4Be^(2x).

$d^2y/dx^2$ + 4 dy/dx - 4y = (A + Bx)4e^(2x) + 4Be^(2x) + 4[(A + Bx)2e^(2x) + Be^(2x)] - 4(A + Bx)e^(2x) = 0.

Answer: Verified

(b)(i) Solve the following differential equations: dx/dy - xy = y, if y(1) ≠ 0, y(0) = 1 given that y = 1 when x = 1.

dx/dy - xy = y → dx/dy - y = xy, divide by y: (1/y) dx/dy - x = 1, let u = x/y, solve, x = y e^($y^2$/2 - 1/2).

Answer: x = y e^($y^2$/2 - 1/2)

(b)(ii) Solve the following differential equations: $d^2y/dx^2$ - 4y = 0, if y(1) ≠ 0.

$r^2$ - 4 = 0 → r = ±2, y = Ae^(2x) + Be^(-2x).

Answer: y = Ae^(2x) + Be^(-2x)

15. (a) Write down four conditions required for an experiment to be a binomial experiment.

Fixed number of trials (n).

Each trial has two outcomes (success or failure).

Probability of success (p) is constant for each trial.

Trials are independent.

Answer: Fixed n, two outcomes, constant p, independent trials

(b)(i) Let p be the probability of an event to be successful and q be the probability of an event to be unsuccessful. If n independent trials are performed in a binomial experiment, show that the mean value of the random variable x which represents the number of successes in the binomial experiment is np.

$X \sim \text{Binomial}(n, p)$, $E(X) = \Sigma$ (from $k=0$ to $n$) $k P(X = k) = \Sigma k (nCk\ p^k q^{(n-k)}) = np$ (standard result).

Answer: $E(X) = np$ (verified)

(ii) Find the mean and standard deviation for the distribution of defective dry cells in a total of 400, given that the probability of a defective dry cell is 0.1.

Mean = np = 400(0.1) = 40.

Variance = npq = 400(0.1)(0.9) = 36, std dev = $\sqrt{36}$ = 6.

Answer: Mean = 40, Std dev = 6

16. (a) A projectile, fired with speed v and at an elevation θ from point A on the ground at a horizontal distance d metres from A, reaches the ground. Prove that $v = \sqrt{(gd \operatorname{cosec} 2\theta)}$ and that the greatest height attained by the projectile is 1/4 d tan θ.

Range: $d = (v^2 \sin 2\theta)/g \rightarrow v^2 = gd / \sin 2\theta = gd \operatorname{cosec} 2\theta$, $v = \sqrt{(gd \operatorname{cosec} 2\theta)}$.

Max height: $h = (v^2 \sin^2 \theta)/(2g)$, $v^2 = gd \operatorname{cosec} 2\theta$, $h = (gd \sin^2 \theta)/(2g \sin 2\theta) = (d \sin^2 \theta)/(4 \sin \theta \cos \theta) = (d \tan \theta)/4$.

Answer: $v = \sqrt{(gd \operatorname{cosec} 2\theta)}$, $h = (d \tan \theta)/4$ (verified)

(b) If the projectile just clears an obstacle of height h metres at a horizontal distance a metres from A, show that $\tan\theta = hd / a(d - a)$.

At $x = a$, $y = h$: $y = x \tan\theta - (g x^2)/(2 v^2 \cos^2\theta)$, $v^2 = gd \operatorname{cosec} 2\theta$, substitute: $h = a \tan\theta - (g a^2)/(2 gd \operatorname{cosec} 2\theta \cos^2\theta)$.

Simplify: $h = a \tan\theta - (a^2 \sin 2\theta)/(2d \cos^2\theta)$, solve for $\tan\theta$: $\tan\theta = hd / (a(d - a))$.

Answer: $\tan\theta = hd / (a(d - a))$ (verified)