THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/1 ADVANCED MATHEMATICS 1

(For Both School and Private Candidates)

Time: 3 Hours ANSWERS Year: 2007

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

1. (a) Simplify the set expression [A' ∩ (A' ∩ B')'] using the laws of algebra of sets.

A' ∩ B' is the set of elements in A' but not in B.

(A' ∩ B')' = (A')' ∪ B [De Morgan's law].

A' ∩ (A')' = ∅ [since A' and (A')' have no elements in common if A* is a subset of A].

A' ∩ (A')' ∪ B = ∅ ∪ (A' ∩ B) = A' ∩ B.

Answer: A' ∩ B

(b) (i) Sets A, B, and C are defined as follows:

$A = \{x \in \mathbb{R} : x < -1 \text{ or } x \geq 2\}$;

$B = \{x \in \mathbb{R} : |x| \leq 2\}$;

$C = \{x \in \mathbb{R} : 1 \leq x \leq 4\}$.

Sketch on the number line the following sets: (A - B) ∪ C.

A: $(-\infty, -1) \cup [2, \infty)$, B: $[-2, 2]$, C: $[1, 4]$.

A - B: Elements in A but not in B = $(-\infty, -2) \cup (2, \infty)$.

(A - B) ∪ C: $(-\infty, -2) \cup (2, \infty) \cup [1, 4] = (-\infty, -2) \cup [1, \infty)$.

Sketch: Number line with an open ray left of -2, and a solid line from 1 to ∞.

Answer: $(-\infty, -2) \cup [1, \infty)$

(b) (ii) Sets A, B, and C are defined as follows:

$A = \{x \in \mathbb{R} : x < -1 \text{ or } x \geq 2\}$;

$B = \{x \in \mathbb{R} : |x| \leq 2\}$;

$C = \{x \in \mathbb{R} : 1 \leq x \leq 4\}$.

Sketch on the number line the following sets: (A ∪ B) ∩ C.

A: $(-\infty, -1) \cup [2, \infty)$, B: $[-2, 2]$, C: $[1, 4]$.

A ∪ B: $(-\infty, \infty)$ [since A and B cover all $\mathbb{R}$].

$(A \cup B) \cap C$: $\mathbb{R} \cap [1, 4] = [1, 4]$.

Sketch: Number line with a solid line from 1 to 4.

Answer: [1, 4]

2. (a) The straight line $2y - x - 16 = 0$ is a perpendicular bisector of the line joining the points A and B. If A is the point (-4, 4), determine the coordinates of B.

Line: $2y - x - 16 = 0 \rightarrow y = (x/2) + 8$, slope = 1/2.

Perpendicular slope: -2.

Midpoint of AB lies on the line. A(-4, 4), let B = (x, y).

Midpoint: $((-4 + x)/2, (4 + y)/2)$, lies on $y = (x/2) + 8 \rightarrow (4 + y)/2 = ((-4 + x)/2)/2 + 8$.

Slope of AB = -2: $(y - 4)/(x + 4) = -2$.

Solve: Midpoint on line $\rightarrow 4 + y = (x - 4)/2 + 16 \rightarrow 4 + y = (x/2) + 14 \rightarrow y = (x/2) + 10$.

Slope: $((x/2) + 10 - 4)/(x + 4) = -2 \rightarrow (x/2 + 6)/(x + 4) = -2 \rightarrow x/2 + 6 = -2x - 8 \rightarrow 5x/2 = -14 \rightarrow x = -28/5$.

$y = (-28/5)/2 + 10 = -14/5 + 10 = 36/5$.

B = (-28/5, 36/5).

Answer: B = (-28/5, 36/5)

2. (b)P moves so that its distance from the origin is always equal to the shortest distance from the line x = 5. Find the equation of the locus.

Distance from origin to P(x, y): $\sqrt{(x^2 + y^2)}$.

Shortest distance from P to x = 5: $|x - 5|$.

$\sqrt{(x^2 + y^2)} = |x - 5|$.

Square both sides: $x^2 + y^2 = (x - 5)^2 \rightarrow x^2 + y^2 = x^2 - 10x + 25 \rightarrow y^2 = -10x + 25 \rightarrow x = (25 - y^2)/10$.

Answer: $x = (25 - y^2)/10$

3. (a) (i) By using the functions $f(x) = x/(1 + x)$, $g(x) = x^2$ and $h(x) = 1/x$, where $x \neq 0$ or $x \neq -1$; show that function composition is associative.

Show $(f \circ g) \circ h = f \circ (g \circ h)$.

$g \circ h(x) = g(h(x)) = g(1/x) = (1/x)^2 = 1/x^2$.

$(f \circ g) \circ h(x) = f(g(h(x))) = f(1/x^2) = (1/x^2)/(1 + 1/x^2) = 1/(x^2 + 1)$.

$f \circ (g \circ h)(x) = f(1/x^2) = (1/x^2)/(1 + 1/x^2) = 1/(x^2 + 1)$.

Both are equal, so associative.

Answer: $(f \circ g) \circ h = f \circ (g \circ h)$, hence associative

3. (a) (ii) By using the functions $f(x) = x/(1 + x)$, $g(x) = x^2$ and $h(x) = 1/x$, where $x \neq 0$ or $x \neq -1$; show that function composition is not commutative.

Show $f \circ g \neq g \circ f$.

$f \circ g(x) = f(g(x)) = f(x^2) = x^2/(1 + x^2)$.

$g \circ f(x) = g(f(x)) = g(x/(1 + x)) = (x/(1 + x))^2 = x^2/(1 + x)^2$.

$x^2/(1 + x^2) \neq x^2/(1 + x)^2$ for general x, so not commutative.

Answer: $f \circ g \neq g \circ f$, hence not commutative

(b) Given that $f(x) = 10x$ and $g(x) = x + 3$, show that $(f \circ g)^{-1}(x) = g^{-1} \circ f^{-1}(x)$.

$f \circ g(x) = f(g(x)) = f(x + 3) = 10(x + 3) = 10x + 30$.

$(f \circ g)^{-1}$: $y = 10x + 30 \rightarrow x = (y - 30)/10$, so $(f \circ g)^{-1}(x) = (x - 30)/10$.

$f^{-1}(x)$: $y = 10x \rightarrow x = y/10$, so $f^{-1}(x) = x/10$.

$g^{-1}(x)$: $y = x + 3 \rightarrow x = y - 3$, so $g^{-1}(x) = x - 3$.

$g^{-1} \circ f^{-1}(x) = g^{-1}(f^{-1}(x)) = g^{-1}(x/10) = (x/10) - 3 = (x - 30)/10$.

Both are equal.

Answer: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

4. (a) Find the value of x given that:

$\log_2(2x + 1) = (4 \log_{x-3} 2 + 1) / (\log_{x-3} 2)$

Simplify the right-hand side:

$(4 \log_{x-3} 2 + 1) / (\log_{x-3} 2) = 4 + (1 / \log_{x-3} 2)$

Recall the logarithmic identity:

$1 / \log_a b = \log_\beta a$

Thus, $1 / \log_{x-3} 2 = \log_2 (x - 3)$

So the equation becomes:

$\log_2 (2x + 1) = 4 + \log_2 (x - 3)$

Move $\log_2 (x - 3)$ to the left-hand side:

$\log_2 (2x + 1) - \log_2 (x - 3) = 4$

Use the logarithmic property: $\log_a M - \log_a N = \log_a (M / N)$

$\log_2 [(2x + 1) / (x - 3)] = 4$

Convert to exponential form:

$(2x + 1) / (x - 3) = 2^4$

$(2x + 1) / (x - 3) = 16$

Multiply both sides by (x - 3):

$2x + 1 = 16(x - 3)$

Expand the right-hand side:

$2x + 1 = 16x - 48$

Move all terms involving x to one side and constants to the other side:

$2x - 16x = -48 - 1$

$-14x = -49$

Solve for x:

$x = -49 / -14$

$x = 3.5$

Thus, the solution is $x = 3.5$.

(b) Find the sum of series:

$5/(1 \times 2 \times 3) + 8/(2 \times 3 \times 4) + 11/(3 \times 4 \times 5) + ... + (3n + 2)/(n(n + 1)(n + 2))$

General term: $(3k + 2)/(k(k + 1)(k + 2))$.

Partial fractions: $(3k + 2)/(k(k + 1)(k + 2)) = A/k + B/(k + 1) + C/(k + 2)$.

Solve: $3k + 2 = A(k + 1)(k + 2) + B\,k(k + 2) + C\,k(k + 1) \rightarrow A = 1/2, B = -2, C = 3/2$.

Term = $(1/2)/k - 2/(k + 1) + (3/2)/(k + 2)$.

Sum: $\Sigma\, [(1/2)/k - 2/(k + 1) + (3/2)/(k + 2)]$ from k = 1 to n.

Telescoping: $(1/2)\,[1 - 1/(n + 1)] + (-2)\,[1/2 - 1/(n + 2)] + (3/2)\,[1/3 - 1/(n + 3)]$.

Simplify: $(n^2 + 3n + 1)/(2(n + 1)(n + 2))$.

Answer: $(n^2 + 3n + 1)/(2(n + 1)(n + 2))$

5. (a) Show that in $\triangle ABC$, $b + c = a \cos \frac{1}{2}(B - C) \operatorname{cosec} \frac{1}{2}A$.

Use the law of sines: $a/\sin A = b/\sin B = c/\sin C = k$.

$b = k \sin B$, $c = k \sin C$, $a = k \sin A$.

$b + c = k (\sin B + \sin C)$.

$\sin B + \sin C = 2 \sin((B + C)/2) \cos((B - C)/2) = 2 \sin(\pi/2 - A/2) \cos((B - C)/2) = 2 \cos(A/2) \cos((B - C)/2)$.

So, $b + c = k (2 \cos(A/2) \cos((B - C)/2))$.

Right side: $a \cos((B - C)/2) \operatorname{cosec}(A/2) = (k \sin A) \cos((B - C)/2) (1/\sin(A/2))$.

$\sin A = 2 \sin(A/2) \cos(A/2)$, so $(\sin A)/(\sin(A/2)) = 2 \cos(A/2)$.

Right side $= k (2 \cos(A/2) \cos((B - C)/2))$.

Both sides match.

Answer: $b + c = a \cos((B - C)/2) \operatorname{cosec}(A/2)$ (verified)

5. (b)Without using tables, find the value of $\tan(\cos^{-1}(1/2) - \tan^{-1}(\sqrt{3}))$.

$\cos^{-1}(1/2) = \pi/3$, $\tan^{-1}(\sqrt{3}) = \pi/3$.

$\cos^{-1}(1/2) - \tan^{-1}(\sqrt{3}) = \pi/3 - \pi/3 = 0$.

$\tan(0) = 0$.

Answer: 0

6. (a)Differentiate $f(x) = 1/(1 + x)$ from first principles.

$f(x) = 1/(1 + x)$, $f(x + h) = 1/(1 + x + h)$.

$f'(x) = \lim(h\to0) [(f(x + h) - f(x))/h] = \lim(h\to0) [(1/(1 + x + h) - 1/(1 + x))/h]$.

$= \lim(h\to0) [((1 + x) - (1 + x + h))/(h (1 + x + h)(1 + x))] = \lim(h\to0) [-h/(h (1 + x + h)(1 + x))] = -1/(1 + x)^2$.

Answer: $-1/(1 + x)^2$

(b) The curve is defined parametrically as $x = 2t + 5t^{-1}$, $y = t^2 + t^{-1}$. Find its gradient at $t = 1$.

$dx/dt = 2 - 5t^{-2}$, $dy/dt = 2t - t^{-2}$.

At $t = 1$: $dx/dt = 2 - 5 = -3$, $dy/dt = 2 - 1 = 1$.

$dy/dx = (dy/dt)/(dx/dt) = 1/(-3) = -1/3$.

Answer: -1/3

7(a)

Find the cosine of the angle between BA and BC where A, B, and C are the points (0, 1, 3), (-1, 0, 1), and (1, -1, -2) respectively.

Vector $BA = A - B = (0 - (-1), 1 - 0, 3 - 1) = (1, 1, 2)$.

Vector $BC = C - B = (1 - (-1), -1 - 0, -2 - 1) = (2, -1, -3)$.

Dot product: $BA \cdot BC = (1)(2) + (1)(-1) + (2)(-3) = 2 - 1 - 6 = -5$.

Magnitudes: $|BA| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$, $|BC| = \sqrt{2^2 + (-1)^2 + (-3)^2} = \sqrt{14}$.

$\cos \theta = (BA \cdot BC) / (|BA|\ |BC|) = -5 / (\sqrt{6}\ \sqrt{14}) = -5 / \sqrt{84} \approx -0.546$.

Answer: $\cos \theta \approx -0.546$

7(b)

Under the action of the forces $E_1 = (i + j + 2k)$ N and $E_2 = (-4i + 6j + 2k)$ N, the body moved a distance $\sqrt{19}$ meters in the direction of the resultant force. Compute the work done on the body correct to two decimal places.

Resultant force $E = E_1 + E_2 = (1 - 4, 1 + 6, 2 + 2) = (-3, 7, 4)$.

Magnitude of $E = \sqrt{(-3)^2 + 7^2 + 4^2} = \sqrt{74}$.

Unit vector in direction of $E = (-3/\sqrt{74}, 7/\sqrt{74}, 4/\sqrt{74})$.

Displacement vector $= (\sqrt{19})\ (-3/\sqrt{74}, 7/\sqrt{74}, 4/\sqrt{74})$.

Work done $= E \cdot \text{displacement} = (-3, 7, 4) \cdot [(\sqrt{19})(-3/\sqrt{74}, 7/\sqrt{74}, 4/\sqrt{74})]$

$= (\sqrt{19}/\sqrt{74})\ [(-3)(-3) + (7)(7) + (4)(4)] = (\sqrt{19}/\sqrt{74})(9 + 49 + 16) = (\sqrt{19}/\sqrt{74})(74) = \sqrt{19} \approx 4.36$.

Answer: 4.36 Joules

8(a)(i)

Find: $\int dx / [(x + 1) \sqrt{x^2 - 1}]$

Let $x = \sec \theta$, $dx = \sec \theta \tan \theta\, d\theta$, $\sqrt{x^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$.

$x + 1 = \sec\theta + 1$.

Integral: $\int (\sec\theta \tan\theta \, d\theta) / [(\sec\theta + 1) \tan\theta] = \int \sec\theta / (\sec\theta + 1) \, d\theta$.

Multiply by $(\sec\theta - 1)/(\sec\theta - 1)$: $\int (\sec^2\theta - \sec\theta) / (\sec^2\theta - 1) \, d\theta = \int (\sec^2\theta - \sec\theta) / \tan^2\theta \, d\theta$.

$= \int (1 - \cos\theta) / \sin^2\theta \, d\theta = \int \text{cosec}^2\theta \, d\theta - \int \text{cosec}\,\theta \cot\theta \, d\theta$.

$= -\cot\theta + \text{cosec}\,\theta + C = -\sqrt{(x^2 - 1)}/x + 1/\sqrt{(x^2 - 1)} + C$.

Answer: $-\sqrt{(x^2 - 1)}/x + 1/\sqrt{(x^2 - 1)} + C$

8(a)(ii)

Find: $\int$ (from -3 to 3) $(x + 3) / (x^2 - 2x - 3) \, dx$

Denominator: $x^2 - 2x - 3 = (x - 3)(x + 1)$.

Partial fractions: $(x + 3) / [(x - 3)(x + 1)] = A/(x - 3) + B/(x + 1)$.

$x + 3 = A(x + 1) + B(x - 3) \rightarrow A = 3/2, B = -1/2$.

Integral: $\int$ (from -3 to 3) $[(3/2)/(x - 3) - (1/2)/(x + 1)] \, dx$.

$= (3/2) \ln|x - 3| - (1/2) \ln|x + 1|$ from -3 to 3.

At $x = 3$: $(3/2) \ln 0 - (1/2) \ln 4$ (undefined).

At $x = -3$: $(3/2) \ln 6 - (1/2) \ln 2 = (3/2) \ln 6 - (1/2) \ln 2$.

Integral diverges due to singularities at $x = -1$ and $x = 3$ within $[-3, 3]$.

Answer: Diverges

(b)Evaluate $\int$ (from 0 to π) $\cos^5 x \sin^2 x \, dx$

$\cos^5 x \sin^2 x = \cos^5 x (1 - \cos^2 x) = \cos^5 x - \cos^7 x$.

Use $\cos x = (1/2)(e^{ix} + e^{-ix})$, but simpler:

Let $u = \sin x$, $du = \cos x \, dx$, limits: $x = 0 \rightarrow u = 0$, $x = \pi \rightarrow u = 0$.

$\cos^4 x = (1 - \sin^2 x)^2$, integral becomes $\int u^2 (1 - u^2)^2 \, du$, but limits make it 0.

Use symmetry: $\int$ (from 0 to π) $= 2 \int$ (from 0 to $\pi/2$) $\cos^5 x \sin^2 x \, dx$.

$\int$ (from 0 to $\pi/2$) $\cos^5 x \sin^2 x \, dx = \int$ (from 0 to $\pi/2$) $\cos^5 x (1 - \cos^2 x) \, dx$.

Use $\cos^5 x = (1/4)(\cos 5x + 5 \cos 3x + 10 \cos x)$, but direct method:

$\int \cos^5 x \sin^2 x\, dx = \int \cos^4 x \cos x \sin^2 x\, dx$, use reduction, final value = 0 (due to symmetry over $[0, \pi]$).

Answer: 0

9(a)(i)

Team A has probability 3/5 of winning whenever it plays. If A plays 4 games, find the probability that A wins at least 1 game.

P(win) = 3/5, P(lose) = 2/5.

P(at least 1 win) = 1 - P(no wins) = $1 - (2/5)^4 = 1 - 16/625 = 609/625$.

Answer: 609/625

9. (a)(ii)Team A has probability 3/5 of winning whenever it plays. If A plays 4 games, find the probability that A wins more than half of the games.

More than half of 4 games = 3 or 4 wins.

Binomial: P(k wins) = C(4,k) (3/5)^k (2/5)^(4-k).

P(3) = C(4,3) $(3/5)^3$ (2/5) = 4 (27/125) (2/5) = 216/625.

P(4) = C(4,4) $(3/5)^4$ = 81/625.

P(3 or 4) = 216/625 + 81/625 = 297/625.

Answer: 297/625

(b) Cherry and Passion are in the table tennis tournament such that the first to win three games wins the tournament. By using tree diagram, how many logic possibilities of the tournament will occur?

Best of 3 wins: Tournament ends when one player wins 3 games.

Possible lengths: 3, 4, or 5 games.

Length 3: CCC, PPP → 2 outcomes.

Length 4: CCCP, CCPC, CPCC, PCCP, PPCP, PCPP → 6 outcomes.

Length 5: CCPCC, CPCCP, PCCCP, CPCPC, CPPCC, PCPCC (and reverse for P) → 10 outcomes.

Total = 2 + 6 + 10 = 18.

Answer: 18 possibilities

10. (a) Below is a frequency distribution table showing the marks obtained by 130 candidates in two different subjects, A and B. Construct a table showing the cumulative frequency distribution in each subject and draw in one diagram the graphs of their ogives.

Subject A:

1-10: 0, 11-20: 0, 21-30: 1, 31-40: 3, 41-50: 6, 51-60: 24, 61-70: 30, 71-80: 31, 81-90: 22, 91-100: 13

Cumulative: 0, 0, 1, 4, 10, 34, 12, 95, 117, 130

Subject B:

1-10: 5, 11-20: 26, 21-30: 30, 31-40: 28, 41-50: 16, 51-60: 9, 61-70: 5, 71-80: 0, 81-90: 1, 91-100: 1

Cumulative: 5, 31, 61, 89, 105, 114, 119, 119, 120, 121

Ogives: Plot cumulative frequencies against upper class boundaries (10, 20, ..., 100) for both subjects on the same graph.

Answer: Cumulative tables as above; ogives plotted

10(b)

From the diagram in (a) above, determine the percentage number of candidates that fail in each subject if the pass mark in subject A is 55 and that in subject B is 35.

Subject A pass mark 55: Cumulative at 50 = 10, total = 130.

Failed = 10, % failed = $(10/130) \times 100 \approx 7.69\%$.

Subject B pass mark 35: Cumulative at 30 = 61, total = 130.

Failed = 61, % failed = $(61/130) \times 100 \approx 46.92\%$.

Answer: A: 7.69%, B: 46.92%

11. In a certain garage, the manager had the following facts: floor space required for a saloon car is 2 m^2 and for a lorry is 3 m^2. Four technicians are required to service a saloon car and three technicians for a lorry per day. He has a maximum of 24 m^2 of floor space and a maximum of 36 technicians available. In addition, he is not allowed to service more lorries than saloon cars. The profit for servicing a saloon car is 40,000/- and a lorry is 60,000/-. How many motor vehicles of each type should be serviced daily in order to maximize the profit?

Let x = saloon cars, y = lorries.

Constraints:

Space: $2x + 3y \leq 24$

Technicians: $4x + 3y \leq 36$

$y \leq x$

$x, y \geq 0$

Profit: $P = 40000x + 60000y$

Intersections:

$2x + 3y = 24$, $y = x \rightarrow 2x + 3x = 24 \rightarrow x = 24/5 = 4.8 \rightarrow (4.8, 4.8)$

$4x + 3y = 36$, $y = x \rightarrow 4x + 3x = 36 \rightarrow x = 36/7 \approx 5.14 \rightarrow (5.14, 5.14)$

$2x + 3y = 24$, $4x + 3y = 36 \rightarrow x = 6$, $y = 4$

Feasible integer points: (0, 0), (0, 8), (6, 4), (4, 4).

Profit: (0, 0) → 0, (0, 8) → 480000, (6, 4) → 480000, (4, 4) → 400000.

Max at (6, 4) or (0, 8), but $y \leq x$ favors (6, 4).

Answer: 6 saloon cars, 4 lorries (Profit = 480000)

12. (a) Find W and Z if W and Z satisfied the system:

$iZ + iW = 2$

$iZ - 2W = -3i$

Rewrite: $iZ + iW = 2 \rightarrow Z + W = -2i$ (multiply by -i)

$iZ - 2W = -3i \rightarrow Z - 2iW = 3$ (multiply by -i)

Subtract: $(Z + W) - (Z - 2iW) = -2i - 3 \rightarrow W + 2iW = -2i - 3 \rightarrow W(1 + 2i) = -2i - 3$

$W = (-2i - 3)/(1 + 2i) = (-2i - 3)(1 - 2i)/(1 + 4) = (-2i - 3 + 4i - 6)/5 = (-9 + 2i)/5$

$Z = -2i - W = -2i - (-9 + 2i)/5 = (-10i + 9 - 2i)/5 = (9 - 12i)/5$

Answer: $W = (-9 + 2i)/5$, $Z = (9 - 12i)/5$

(b) Express $Z = 1 - i$ in polar form. Hence, find the two complex values of W if $W^2 = Z$, leaving your answer in polar form.

$Z = 1 - i$, $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$, $\theta = \tan^{-1}(-1/1) = -\pi/4$

$Z = \sqrt{2} (\cos(-\pi/4) + i \sin(-\pi/4))$

$W^2 = Z \rightarrow W = Z^{(1/2)}$, $r^{(1/2)} = (\sqrt{2})^{(1/2)} = 2^{(1/4)}$, $\theta/2 = -\pi/8, -\pi/8 + \pi$

$W = 2^{(1/4)} (\cos(-\pi/8) + i \sin(-\pi/8)), 2^{(1/4)} (\cos(7\pi/8) + i \sin(7\pi/8))$

Answer: $Z = \sqrt{2} (\cos(-\pi/4) + i \sin(-\pi/4))$, $W = 2^{(1/4)} (\cos(-\pi/8) + i \sin(-\pi/8)), 2^{(1/4)} (\cos(7\pi/8) + i \sin(7\pi/8))$

(c) Write the complex number (3 - i)/(1 + 2i) in polynomial form.

$(3 - i)/(1 + 2i) = (3 - i)(1 - 2i)/(1 + 4) = (3 - 6i - i + 2i^2)/5 = (3 - 2 - 7i)/5 = (1 - 7i)/5$

Answer: $(1 - 7i)/5$

13. (a) Show that the equation $e^x + 3 - x$ has roots in the interval [0, 1]. Also, find this root correct to two decimal places in three iterations by using Regula Falsi method.

$f(x) = e^x + 3 - x$, $f(0) = e^0 + 3 - 0 = 4 > 0$, $f(1) = e^1 + 3 - 1 \approx 2.718 + 2 \approx 4.718 - 1 = 3.718 > 0$.

Check $f(0.5) = e^{0.5} + 3 - 0.5 \approx 1.649 + 2.5 \approx 4.149 > 0$ (no sign change, try wider interval).

Try [-1, 1]: $f(-1) = e^{(-1)} + 3 - (-1) \approx 0.368 + 4 \approx 4.368 > 0$, still no sign change.

Try [1, 2]: $f(2) = e^2 + 3 - 2 \approx 7.389 + 1 \approx 8.389 > 0$ (no root in [0, 1], likely typo in problem).

Assume problem meant $e^x - 3 - x$: $f(x) = e^x - 3 - x$, $f(0) = 1 - 3 = -2$, $f(1) = e - 4 \approx -1.282 < 0$, $f(2) = e^2 - 5 \approx 2.389 > 0$.

Root in [1, 2].

Regula Falsi: $x1 = 1 - (-1.282)(2 - 1)/(2.389 - (-1.282)) \approx 1 + 1.282/3.671 \approx 1.349$, $f(1.349) \approx -0.615$.

$x2 = 1.349 - (-0.615)(2 - 1.349)/(2.389 - (-0.615)) \approx 1.349 + 0.615(0.651)/3.004 \approx 1.482$, $f(1.482) \approx -0.237$.

$x3 = 1.482 - (-0.237)(2 - 1.482)/(2.389 - (-0.237)) \approx 1.482 + 0.237(0.518)/2.626 \approx 1.529 \approx 1.53$.

Answer: Root in [1, 2], $x \approx 1.53$ (assuming $e^x - 3 - x$)

(b) Use trapezoidal rule and nine ordinates to obtain an approximate value of the definite integral: $\int$ (from 0 to π) $(x \sin x) / (1 + \cos^2 x)\, dx$

$h = \pi/8$, $x = 0, \pi/8, ..., \pi$

$f(x) = (x \sin x)/(1 + \cos^2 x)$, values: 0, 0.076, 0.149, 0.219, 0.285, 0.347, 0.404, 0.456, 0.502

Trapezoidal: $(h/2) [f(0) + 2(f(\pi/8) + ... + f(7\pi/8)) + f(\pi)]$

$= (\pi/16) [0 + 2(0.076 + 0.149 + 0.219 + 0.285 + 0.347 + 0.404 + 0.456) + 0.502] \approx 1.234$

Answer: 1.234

14. (a)Solve the differential equation $\cot x\, dy/dx = 1 - y^2$ given that $y = 0$ when $x = \pi/4$.

$\cot x\, dy/dx = 1 - y^2 \rightarrow dy/(1 - y^2) = dx/\tan x = dx \cot x / \sin^2 x$.

$\int dy/(1 - y^2) = \int \cot x\, dx$, left: $\tanh^{-1} y$, right: $\ln|\sin x| + C$.

$\tanh^{-1} y = \ln|\sin x| + C$, at $x = \pi/4$, $y = 0$: $0 = \ln(\sqrt{2}/2) + C \rightarrow C = -\ln(\sqrt{2}/2) = \ln(\sqrt{2})$.

$\tanh^{-1} y = \ln|\sin x| + \ln(\sqrt{2}) = \ln(\sqrt{2} \sin x)$.

$y = \tanh(\ln(\sqrt{2} \sin x))$.

Answer: $y = \tanh(\ln(\sqrt{2} \sin x))$

(b)(i) The rate from which the atoms in a mass of radioactive material are disintegrating is proportional to N, the number of atoms present at any time. Initially the number of atoms was M. Form and solve the differential equation that represents this data.

$dN/dt = -kN$ ($k > 0$, decay), $dN/N = -k\, dt$.

$\int dN/N = \int -k\, dt \rightarrow \ln N = -kt + C$.

At $t = 0$, $N = M$: $\ln M = C$, so $\ln N = -kt + \ln M \rightarrow N = M e^{(-kt)}$.

Answer: $N = M e^{(-kt)}$

(ii) Given that half of the original mass disintegrates in 152 days, evaluate the constant of proportionality in the differential equation.

$N = M/2$ when $t = 152$: $M/2 = M e^{(-k(152))} \rightarrow 1/2 = e^{(-152k)}$.

$\ln(1/2) = -152k \rightarrow -\ln 2 = -152k \rightarrow k = \ln 2 / 152 \approx 0.00456$.

Answer: $k \approx 0.00456$

(iii)Sketch the graph to represent the number of atoms N at any time t.

N = M e^(-kt), $k \approx 0.00456$.

At t = 0, N = M; as $t \to \infty$, $N \to 0$.

Exponential decay curve: starts at (0, M), decreases, half-life at t = 152 days.

Answer: Exponential decay curve from (0, M)

15. (a)Show that Var E(x) = $E(x^2) - [E(x)]^2$

$Var(X) = E[(X - E(X))^2] = E(X^2 - 2X E(X) + [E(X)]^2) = E(X^2) - 2 E(X) E(X) + [E(X)]^2 = E(X^2) - [E(X)]^2$.

Answer: $Var(X) = E(X^2) - [E(X)]^2$ (verified)

(b) The amount of sulphur oxide produced by an industrial plant in 80 days is as shown in the following table. Calculate the interquartile range for this distribution.

Tons: 5-8.9 (3), 9-12.9 (10), 13-16.9 (14), 17-20.9 (25), 21-24.9 (15), 25-28.9 (9), 29-32.9 (4)

Total = 80.

Q1 (20th): Cumulative: 3, 13, 27 → 13-16.9 class, $Q1 = 13 + (20 - 13)/14 \times 4 \approx 15$.

Q3 (60th): Cumulative: 52, 67 → 21-24.9 class, $Q3 = 21 + (60 - 52)/15 \times 4 \approx 23.13$.

$IQR = Q3 - Q1 \approx 23.13 - 15 \approx 8.13$.

Answer: $IQR \approx 8.13$

16. (a) Three forces are exerted on an object as follows: E_1 = 5 units to the right, E_2 = 10 units upwards, and E_3 = 2 units inclined at an angle of 30° to the horizontal. Find a single force equivalent to the three forces acting together.

$E_1 = (5, 0)$, $E_2 = (0, 10)$, $E_3 = (2 \cos 30°, 2 \sin 30°) = (\sqrt{3}, 1)$.

Resultant = $(5 + \sqrt{3}, 10 + 1) = (5 + \sqrt{3}, 11)$.

Magnitude = $\sqrt{(5 + \sqrt{3})^2 + 11^2} \approx 12.28$, angle = $\tan^{-1}(11/(5 + \sqrt{3})) \approx 66°$.

Answer: $(5 + \sqrt{3}, 11)$, or magnitude ≈ 12.28 at 66° to horizontal (b)A mouth of the gun is inclined at an angle of 15° to the horizontal. The gun is fired from ground level with an initial speed of 150 m/s. Assuming that the gun is fired at the origin of the xy plane, determine the equation for the path of the bullet. (Take $g = 10\ m/s^2$)

Initial velocity: $u_x = 150 \cos 15° \approx 144.89$, $u_y = 150 \sin 15° \approx 38.82$.

$x = u_x t = 144.89\ t$, $y = u_y t - (1/2) g t^2 = 38.82\ t - 5 t^2$.

$t = x/144.89$, $y = 38.82 (x/144.89) - 5 (x/144.89)^2 \approx 0.268 x - 0.000238 x^2$.

Answer: $y \approx 0.268 x - 0.000238 x^2$