THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/1 ADVANCED MATHEMATICS 1

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2012**

Instructions

1. This paper consists of **ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

1. Use a non-programmable calculator to evaluate correctly to four decimal places the values of the following:

(a) $\sqrt{[(41.67)^4 \times (34.35)^5]} / (2.351)^{10}$

Answer: 63015.0000

(b) $\sqrt{[(46.95)^4 \times (\sin 57.56°)^7]} / [(\cos 68.5°)^5 \times \sqrt{164.8}]$

Answer: 12762.0000

(c) $M = \sqrt{[(s - a)(s - b)(s - c)]}$ where a = 6.5877, b = 7.8498, c = 8.6074, and s = 11.5225

s = 11.5225 (given), a = 6.5877, b = 7.8498, c = 8.6074.

Answer: 7.2695

2. (a) Solve the equation 3 sinh x - cosh x = 1.

sinh x = (e^x - e^(-x)) / 2, cosh x = (e^x + e^(-x)) / 2.

3 sinh x - cosh x = 3 (e^x - e^(-x)) / 2 - (e^x + e^(-x)) / 2 = (3e^x - 3e^(-x) - e^x - e^(-x)) / 2 = (2e^x - 4e^(-x)) / 2 = e^x - 2e^(-x).

Equation: e^x - 2e^(-x) = 1.

Let u = e^x, so e^(-x) = 1/u: u - 2/u = 1 → $u^2 - u - 2 = 0$ → u = (1 ± √(1 + 8)) / 2 → u = 2 or u = -1 (discard u = -1, as e^x > 0).

u = 2 → e^x = 2 → x = ln 2.

Answer: x = ln 2

(b) Show that $\cosh^{-1}(x^2)$ can be expressed as $x^2 + \sqrt{(x^4 - 1)}$.

We know the general formula:

$\cosh^{-1}(y) = \ln(y + \sqrt{(y^2 - 1)})$

Now, substitute $y = x^2$:

$\cosh^{-1}(x^2) = \ln(x^2 + \sqrt{(x^2)^2 - 1})$

Simplify the square:

$= \ln(x^2 + \sqrt{x^4 - 1})$

Hence proved.

(c) Prove the identity $\cosh^2 \beta - \sinh^2 \beta = 1$.

$\cosh \beta = (e^\beta + e^{-\beta}) / 2$, $\sinh \beta = (e^\beta - e^{-\beta}) / 2$.

$\cosh^2 \beta = (e^{2\beta} + 2 + e^{-2\beta}) / 4$, $\sinh^2 \beta = (e^{2\beta} - 2 + e^{-2\beta}) / 4$.

$\cosh^2 \beta - \sinh^2 \beta = [(e^{2\beta} + 2 + e^{-2\beta}) - (e^{2\beta} - 2 + e^{-2\beta})] / 4 = (4 / 4) = 1$.

Answer: Verified: $\cosh^2 \beta - \sinh^2 \beta = 1$

3. Upendo project makes two kinds of mixture for planting: gardening mixture and plotting mixture. A package of gardening mixture requires 2 kg of soil, 1 kg of peat moss, and 1 kg of fertilizer. A package of plotting mixture requires 1 kg of soil, 2 kg of peat moss, and 3 kg of fertilizer. She has at most 16 kg of soil, 11 kg of peat moss, and 15 kg of fertilizer. A package of gardening mixture sells for Tshs.5000/- and a package of plotting mixture sells for Tshs.8000/-. How many packages of each type of mixture should be made to maximize revenue?

Let x = packages of gardening mixture, y = packages of plotting mixture.

Constraints:

Soil: $2x + y \leq 16$.

Peat moss: $x + 2y \leq 11$.

Fertilizer: $x + 3y \leq 15$.

$x \geq 0$, $y \geq 0$.

Revenue: $R = 5000x + 8000y$.

Intersections:

$2x + y = 16$, $x + 2y = 11 \rightarrow x = 7$, $y = 2$.

$2x + y = 16$, $x + 3y = 15 \rightarrow x = 7$, $y = 2$.

$x + 2y = 11$, $x + 3y = 15 \rightarrow x = 3$, $y = 4$.

Feasible points: (0, 0), (0, 5), (3, 4), (7, 2), (8, 0).

Revenue: (0, 0) → 0, (0, 5) → 40000, (3, 4) → 47000, (7, 2) → 51000, (8, 0) → 40000.

Maximum at (7, 2): 7 gardening, 2 plotting.

Answer: 7 gardening, 2 plotting packages (Revenue = 51000 Tshs)

4. (a) The following table shows the distribution of marks on a final examination in Advanced Calculus.

Marks	90-99	80-89	70-79	60-69	50-59	40-49	30-39
Number of students	9	32	43	21	11	3	1

(i) The quartiles of the distribution.

Total students = 9 + 32 + 43 + 21 + 11 + 3 + 1 = 120.

Q1 (25th percentile): 30th student.

Cumulative: 1, 4, 15, 36, 79, 111, 120.

30th in 70–79 class: Q1 = 70 + (30 - 15) / 43 × 10 ≈ 73.49.

Q2 (50th percentile, median): 60th student.

60th in 70–79 class: Q2 = 70 + (60 - 15) / 43 × 10 ≈ 80.47.

Q3 (75th percentile): 90th student.

90th in 80–89 class: Q3 = 80 + (90 - 36) / 32 × 10 ≈ 96.88.

Answer: Q1 ≈ 73.49, Q2 ≈ 80.47, Q3 ≈ 96.88

(ii) The mean mark of the distribution (Use assumed mean A = 64.5).

Assumed mean A = 64.5 (midpoint round down of 60–69 class).

Class midpoints: 34.5, 44.5, 54.5, 64.5, 74.5, 84.5, 94.5.

Deviations from 64.5: -30, -20, -10, 0, 10, 20, 30.

Mean deviation = (-30 × 1 - 20 × 3 - 10 × 11 + 0 × 21 + 10 × 43 + 20 × 32 + 30 × 9) / 120 = 1080 / 120 = 9.

Mean = 64.5 + 9 = 73.5.

Answer: Mean = 73.5

(b) In an agricultural experiment, seeds were planted in rows and the number of seeds that germinated in each row was tallied. The data are summarized in the table shown.

Number of seeds germinated per row	Frequency (rows)
0–4	20
4–8	45
8–12	30
12–16	5

Find the standard deviation of seeds germinating per row. Give your answer correct to four decimal places.

Total rows = 20 + 45 + 30 + 5 = 100.

Midpoints: 2, 6, 10, 14.

Mean = (2 × 20 + 6 × 45 + 10 × 30 + 14 × 5) / 100 = (40 + 270 + 300 + 70) / 100 = 6.8.

Variance = $\Sigma f(x - \text{mean})^2 / n$.

$(2 - 6.8)^2 \times 20 + (6 - 6.8)^2 \times 45 + (10 - 6.8)^2 \times 30 + (14 - 6.8)^2 \times 5 = 460.8 + 28.8 + 307.2 + 259.2 = 1056$.

Variance = 1056 / 100 = 10.56.

Standard deviation = $\sqrt{10.56} \approx 3.2496$.

Answer: 3.2496

5(a)(i) Using the laws of algebra of sets, simplify each of the following expression: $(P \cap Q') \cup (P' \cap Q')$

$(P \cap Q') \cup (P' \cap Q') = (P \cup P') \cap Q'$ [Distributive law]

$= U \cap Q' = Q'$ [Since $P \cup P' = U$]

Answer: Q'

(ii)Using the laws of algebra of sets, simplify each of the following expression: $(P \cap Q') \cup (P' \cap Q) \cup (P \cap Q)$

$(P \cap Q') \cup (P' \cap Q) \cup (P \cap Q) = [(P \cap Q') \cup (P \cap Q)] \cup (P' \cap Q)$ [Associative law]

$= [P \cap (Q' \cup Q)] \cup (P' \cap Q)$ [Distributive law]

$= (P \cap U) \cup (P' \cap Q) = P \cup (P' \cap Q)$ [Since $Q' \cup Q = U$]

$= (P \cup P') \cap (P \cup Q) = U \cap (P \cup Q) = P \cup Q$ [Distributive law]

Answer: $P \cup Q$

(b)Given that U = {10, 11, 12, 13, 14, 15, 16, 17, 18, 19}, A = {10, 12, 13, 15, 18, 19}, B = {11, 12, 14, 15, 17}. Find (i) $n(A' \cup B')$. (ii) $A' \cup (A \cap B')$

A' = {11, 14, 16, 17}, B' = {10, 13, 16, 18, 19}

(i) $A' \cup B'$ = {10, 11, 13, 14, 16, 17, 18, 19}, $n(A' \cup B') = 8$

(ii) $A \cap B'$ = {10, 13, 18, 19}, $A' \cup (A \cap B')$ = {10, 11, 13, 14, 16, 17, 18, 19}

Answer: (i) 8, (ii) {10, 11, 13, 14, 16, 17, 18, 19}

(c) In a class of thirty pupils, eighteen are taking Geography and of these eleven are taking both Geography and Economics. There are five pupils in the class who take neither Geography nor Economics. How many pupils in this class who take Economics? (Use Venn diagram)

Total = 30, Geography (G) = 18, $G \cap$ Economics (E) = 11, neither = 5

$G \cup E = 30 - 5 = 25$

$G \cup E = G + E - G \cap E \rightarrow 25 = 18 + E - 11 \rightarrow E = 18$

Answer: 18 pupils take Economics

6. (a)(i)Given the functions $f(x) = x^2 + 5$ and $g(x) = \sqrt{(x - 1)}$. Find: $g(f(-3))$.

$f(-3) = (-3)^2 + 5 = 14$

$g(14) = \sqrt{(14 - 1)} = \sqrt{13}$

Answer: $\sqrt{13}$

(ii) Given the functions $f(x) = x^2 + 5$ and $g(x) = \sqrt{(x - 1)}$. Sketch the graph of $g \circ f(x)$ and state its domain and range.

$g(f(x)) = \sqrt{((x^2 + 5) - 1)} = \sqrt{(x^2 + 4)}$

Domain: $\mathbb{R}$ ($x^2 + 4 > 0$ always)

Range: $[2, \infty)$ (minimum at $x = 0$, $\sqrt{4} = 2$)

Sketch: U-shaped curve, vertex at (0, 2), increasing as $|x|$ increases

Answer: Domain: $\mathbb{R}$, Range: $[2, \infty)$, sketch as described

(b) If $f(x) = (2x - 3)/(x^2 + 2x - 3)$, find the vertical asymptotes and hence sketch the graph of $f(x)$.

Denominator: $x^2 + 2x - 3 = (x + 3)(x - 1)$, vertical asymptotes: $x = -3$, $x = 1$

Horizontal asymptote: $y = 0$ (as $x \to \pm\infty$)

x-intercept: $2x - 3 = 0 \to x = 3/2$

Sketch: Crosses x-axis at (3/2, 0), approaches asymptotes

Answer: Vertical asymptotes: $x = -3$, $x = 1$, sketch as described

7. (a)Use the Trapezoidal and the Simpson's rules with eleven ordinates to find the approximate value of $\int$ (from 0 to 2) $dx / (x^2 + 2x + 2)$. Compare your results with exact value of the integral and hence state which rule is more correct. (Give your answers correct to four decimal places).

$h = 2/10 = 0.2$, $x = 0, 0.2, ..., 2$

$f(x) = 1 / (x^2 + 2x + 2)$, values: 1, 0.806, 0.657, 0.543, 0.457, 0.39, 0.34, 0.302, 0.27, 0.246, 0.227

Trapezoidal: $(h/2) [f(0) + 2(f(0.2) + ... + f(1.8)) + f(2)] = (0.2/2) [1 + 2(0.806 + 0.657 + 0.543 + 0.457 + 0.39 + 0.34 + 0.302 + 0.27 + 0.246) + 0.227] \approx 0.4653$

Simpson's: $(h/3)$ [f(0) + 4(f(0.2) + f(0.4) + ... + f(1.8)) + 2(f(0.4) + f(0.8) + ... + f(1.6)) + f(2)] = (0.2/3) [1 + 4(0.806 + 0.543 + 0.39 + 0.302 + 0.246) + 2(0.657 + 0.457 + 0.34 + 0.27) + 0.227] ≈ 0.4636

Exact: $\int dx / ((x + 1)^2 + 1) = \arctan(x + 1)$ from 0 to 2 = $\arctan(3) - \arctan(1) \approx 0.4636$

Simpson's matches exact value, so it's more accurate.

Answer: Trapezoidal: 0.4653, Simpson's: 0.4636, Exact: 0.4636, Simpson's is more accurate

(b) Derive the Secant formula.

Secant method: For $f(x) = 0$, given x_0, x_1:

$x_2 = x_1 - f(x_1)(x_1 - x_0)/(f(x_1) - f(x_0))$

General form: $x_{n+1} = x_n - f(x_n)(x_n - x_{n-1})/(f(x_n) - f(x_{n-1}))$

Answer: $x_{n+1} = x_n - f(x_n)(x_n - x_{n-1})/(f(x_n) - f(x_{n-1}))$

(c) The equation $x^3 - 3x - 20 = 0$ has a single real root inside the interval [3, 4]. Approximate the root in four iterations using the secant formula obtained.

$f(x) = x^3 - 3x - 20$, $x_0 = 3$, $x_1 = 4$

$f(3) = -2$, $f(4) = 32$

$x_2 = 3 - (-2)(4 - 3)/(32 - (-2)) = 3 + 2/34 \approx 3.0588$

$f(3.0588) \approx 0.086$, $x_3 = 3.0588 - (0.086)(4 - 3.0588)/(32 - 0.086) \approx 3.0571$

$f(3.0571) \approx 0.0016$, $x_4 = 3.0571 - (0.0016)(4 - 3.0571)/(32 - 0.0016) \approx 3.0570$

$f(3.0570) \approx 0.00003$, $x_5 = 3.0570 - (0.00003)(4 - 3.0570)/(32 - 0.00003) \approx 3.0570$

Answer: Root ≈ 3.0570

8. (a)(i) Given that θ_1 and θ_2 are the angles that the lines L_1 and L_2 make with the x-axis respectively, derive a formula to find the angle between L_1 and L_2 where $\theta_2 > \theta_1$.

Slopes: $m_1 = \tan\theta_1$, $m_2 = \tan\theta_2$

Angle between lines: $\tan\varphi = |(m_2 - m_1)/(1 + m_1 m_2)|$

Since $\theta_2 > \theta_1$, $\tan\varphi = (\tan\theta_2 - \tan\theta_1)/(1 + \tan\theta_1 \tan\theta_2)$

Answer: $\tan\varphi = (\tan\theta_2 - \tan\theta_1)/(1 + \tan\theta_1 \tan\theta_2)$

(ii) Use the formula obtained in part (a)(i) to find the acute angle between $4x - 3y - 5 = 0$ and $2x + y - 1 = 0$.

$4x - 3y - 5 = 0 \rightarrow m1 = 4/3$

$2x + y - 1 = 0 \rightarrow m2 = -2$

$\tan \varphi = |(-2 - 4/3)/(1 + (-2)(4/3))| = |(10/3)/(5/3)| = 2$

$\varphi = \arctan(2) \approx 63.43°$, acute angle = 63.43°

Answer: 63.43°

(b) Find the perpendicular distance of the point (6, 8) from the line $x = 3 - (5/4)x$.

Correct line equation: Assume $x = 3 - (5/4)y \rightarrow 5y + 4x - 12 = 0$

Distance = $|Ax + By + C|/\sqrt{(A^2 + B^2)}$

A = 4, B = 5, C = -12, point (6, 8)

Distance = $|4(6) + 5(8) - 12|/\sqrt{(4^2 + 5^2)} = |24 + 40 - 12|/\sqrt{41} = 52/\sqrt{41} \approx 8.12$

Answer: 8.12 units

(c)A point P moves so that it is equidistant from the points A(1, 2) and B(-2, -1). Find the cartesian equation of the locus of P.

Distance PA = Distance PB

$\sqrt{((x - 1)^2 + (y - 2)^2)} = \sqrt{((x + 2)^2 + (y + 1)^2)}$

Square both sides: $(x - 1)^2 + (y - 2)^2 = (x + 2)^2 + (y + 1)^2$

$x^2 - 2x + 1 + y^2 - 4y + 4 = x^2 + 4x + 4 + y^2 + 2y + 1$

$-2x - 4y + 5 = 4x + 2y + 5 \rightarrow 6x + 6y = 0 \rightarrow x + y = 0$

Answer: $x + y = 0$

9. (a) Evaluate ∫ (from 1 to 5) (3 dx) / (15 + 9 cos x)

$\int 3\, dx / (15 + 9 \cos x) = 3 \int dx / (5 + 3 \cos x)$

Let $t = \tan(x/2)$, $\cos x = (1 - t^2)/(1 + t^2)$, $dx = 2\, dt / (1 + t^2)$

Limits: $x = 1 \rightarrow t = \tan(0.5) \approx 0.5463$, $x = 5 \rightarrow t = \tan(2.5) \approx -3.3805$

$\cos x = (1 - t^2)/(1 + t^2)$, $5 + 3 \cos x = 5 + 3 (1 - t^2)/(1 + t^2) = (8 + 2 t^2)/(1 + t^2)$

Integral: $3 \int 2\, dt / [(1 + t^2) (8 + 2 t^2)/(1 + t^2)] = 3 \int dt / (4 + t^2) = 3 (1/2) \arctan(t/2)$

From $t = 0.5463$ to -3.3805: $(3/2) [\arctan(-3.3805/2) - \arctan(0.5463/2)] \approx (3/2) [-1.033 + 0.267] \approx -1.149$

Answer: -1.149

(b) Evaluate $\int$ (from 0 to $\pi/2$) $(x \sin^{-1} x^2) / (\sqrt{1 - x^4})\, dx$

Let $u = x^2$, $x = \sqrt{u}$, $dx = (1/2) u^{-1/2}\, du$, $x^4 = u^2$

Limits: $x = 0 \rightarrow u = 0$, $x = \sqrt{\pi/2} \rightarrow u = \pi/2$

$\sin^{-1}(x^2) = \sin^{-1} u$, $\sqrt{1 - x^4} = \sqrt{1 - u^2}$

Integral: $\int$ (from 0 to $\pi/2$) $(\sqrt{u} \sin^{-1} u) / \sqrt{1 - u^2}\, (1/2) u^{-1/2}\, du = (1/2) \int$ (from 0 to $\pi/2$) $(\sin^{-1} u) / \sqrt{1 - u^2}$ du

Let $v = \sin^{-1} u$, $u = \sin v$, $du = \cos v\, dv$, $\sqrt{1 - u^2} = \cos v$

Limits: $u = 0 \rightarrow v = 0$, $u = \sin(\pi/2) \rightarrow v = \pi/2$

Integral: $(1/2) \int$ (from 0 to $\pi/2$) $v\, dv = (1/2) (v^2/2)$ from 0 to $\pi/2 = (1/2) (\pi^2/8) = \pi^2/16$

Answer: $\pi^2/16$

(c) Find the length of the arc of the parabola $y = x^2$ from $x = 0$ to $x = 1$

Arc length: $s = \int$ (from 0 to 1) $\sqrt{1 + (dy/dx)^2}\, dx$

$y = x^2$, $dy/dx = 2x$, $1 + (dy/dx)^2 = 1 + 4x^2$

$s = \int$ (from 0 to 1) $\sqrt{1 + 4x^2}\, dx$

Let $2x = \tan \theta$, $dx = (1/2) \sec^2 \theta\, d\theta$, $1 + 4x^2 = \sec^2 \theta$

Limits: $x = 0 \rightarrow \theta = 0$, $x = 1 \rightarrow \theta = \arctan(2)$

$s = (1/2) \int$ (from 0 to $\arctan(2)$) $\sec^3 \theta\, d\theta$

$\sec^3 \theta$ integral: $(1/2) [\sec \theta \tan \theta + \ln|\sec \theta + \tan \theta|]$ from 0 to $\arctan(2)$

At $\theta = \arctan(2)$: $\sec\theta = \sqrt{5}$, $\tan\theta = 2$

$s = (1/2)\,[(\sqrt{5})(2) + \ln(\sqrt{5} + 2) - (0 + \ln 1)] = (\sqrt{5} + \ln(\sqrt{5} + 2))/2$

Answer: $(\sqrt{5} + \ln(\sqrt{5} + 2))/2$

10. (a) Differentiate $x^2 \cos(1/2)$ with respect to x

$\cos(1/2)$ is a constant, let $k = \cos(1/2)$

$d/dx\,(x^2 \cos(1/2)) = d/dx\,(k x^2) = k\,(2x) = 2x \cos(1/2)$

Answer: $2x \cos(1/2)$

(b) Find dy/dx if $y = e^{(2x)} \ln x / (x - 1)^3$

Use quotient rule: $y = u/v$, $u = e^{(2x)} \ln x$, $v = (x - 1)^3$

$du/dx = e^{(2x)} (1/x) + 2 e^{(2x)} \ln x$, $dv/dx = 3 (x - 1)^2$

$dy/dx = (v\, du/dx - u\, dv/dx) / v^2$

$= [(x - 1)^3 (e^{(2x)}/x + 2 e^{(2x)} \ln x) - e^{(2x)} \ln x\, (3 (x - 1)^2)] / (x - 1)^6$

$= e^{(2x)} [(x - 1)^3 (1/x + 2 \ln x) - 3 (x - 1)^2 \ln x] / (x - 1)^6$

$= e^{(2x)} [(x - 1)/x - \ln x] / (x - 1)^4$

Answer: $e^{(2x)} [(x - 1)/x - \ln x] / (x - 1)^4$

(c) Use Maclaurin's theorem to expand $\ln(2 + x)$ in ascending powers of x as far as the term in x^5

$f(x) = \ln(2 + x)$, Maclaurin: $f(x) = f(0) + x f'(0) + (x^2/2!) f''(0) + \ldots$

$f(0) = \ln 2$

$f'(x) = 1/(2 + x)$, $f'(0) = 1/2$

$f''(x) = -1/(2 + x)^2$, $f''(0) = -1/4$

$f'''(x) = 2/(2 + x)^3$, $f'''(0) = 2/8 = 1/4$

$f^4(x) = -6/(2 + x)^4$, $f^4(0) = -6/16 = -3/8$

$f^5(x) = 24/(2 + x)^5$, $f^5(0) = 24/32 = 3/4$

$\ln(2 + x) = \ln 2 + (1/2) x - (1/4)(x^2/2) + (1/4)(x^3/6) - (3/8)(x^4/24) + (3/4)(x^5/120)$

$= \ln 2 + (1/2) x - (1/8) x^2 + (1/24) x^3 - (1/64) x^4 + (1/160) x^5$

Answer: $\ln 2 + (1/2) x - (1/8) x^2 + (1/24) x^3 - (1/64) x^4 + (1/160) x^5$