THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/1 ADVANCED MATHEMATICS 1

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2013**

Instructions

1. This paper consists of **ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Using a non-programmable calculator evaluate:

(i) $\sqrt{(0.38452 / 124.8)} / [(0.04382)^2 / (\sqrt{(431)})]^7$ correct to seven significant figures.

Answer: 1.033400×10^{25}

(ii) Σ (from k=1 to 4) [ln(1 + 1/k)], giving your answer to three significant figures.

Answer: 0.916

(b) Cosmic-ray bombardment in the atmosphere produces neutrons, which in turn react with nitrogen to produce radioactive Carbon-14 (^{14}C). Radioactive ^{14}C enters all living tissues through Carbon dioxide, which is first absorbed by plants. As long as a plant or animal is alive, ^{14}C is maintained in the living organism at a constant level. Once the organism dies, however, ^{14}C decays according to the equation: $A = A_0 e^{(-kt)}$, where A is the amount present after t years and A_0 is the amount present at time t = 0. If 500 milligrams of ^{14}C are present at the start, use a calculator to:

(i) Find how many milligrams will be present in 15,000 years correct to two decimal places.

Answer: 81.50 mg

(ii) Calculate the number of years it takes for 1.89 milligrams to remain.

$1.89 = 500 e^{(-0.00012097 t)}$.

Answer: 46110 years

2. (a) If 2 cosh y - 7 sinh x = 3 and cosh y - 3 sinh x = 2, find the real values of x and y in logarithmic form that satisfy the two equations.

Equations:

2 cosh y - 7 sinh x = 3 (1).

cosh y - 3 sinh x = 2 (2).

From (2): cosh y = 3 sinh x + 2.

Substitute into (1): 2(3 sinh x + 2) - 7 sinh x = 3 → 6 sinh x + 4 - 7 sinh x = 3 → -sinh x = -1 → sinh x = 1.

sinh x = 1 → x = $\ln(1 + \sqrt{2})$.

$\cosh y = 3(1) + 2 = 5 \rightarrow y = \ln(5 + \sqrt{24}) = \ln(5 + 2\sqrt{6})$.

Answer: $x = \ln(1 + \sqrt{2})$, $y = \ln(5 + 2\sqrt{6})$

2. (b) (i) Verify that $\cosh 5x + \cosh 3x - 2 \cosh x = 16 \sinh^2 x \cosh^3 x$.

Left side: $\cosh 5x + \cosh 3x - 2 \cosh x$.

Use identity: $\cosh a + \cosh b = 2 \cosh((a + b)/2) \cosh((a - b)/2)$.

$\cosh 5x + \cosh 3x = 2 \cosh((5x + 3x)/2) \cosh((5x - 3x)/2) = 2 \cosh 4x \cosh x$.

So, left side $= 2 \cosh 4x \cosh x - 2 \cosh x = 2 \cosh x (\cosh 4x - 1)$.

$\cosh 4x = 2 \cosh^2 2x - 1$, $\cosh 2x = 2 \cosh^2 x - 1$.

$\cosh 4x = 2 (2 \cosh^2 x - 1)^2 - 1 = 2 (4 \cosh^4 x - 4 \cosh^2 x + 1) - 1 = 8 \cosh^4 x - 8 \cosh^2 x + 1$.

$\cosh 4x - 1 = 8 \cosh^4 x - 8 \cosh^2 x$.

Left side $= 2 \cosh x (8 \cosh^4 x - 8 \cosh^2 x) = 16 \cosh^5 x - 16 \cosh^3 x$.

Right side: $16 \sinh^2 x \cosh^3 x$.

Use $\cosh^2 x - \sinh^2 x = 1 \rightarrow \sinh^2 x = \cosh^2 x - 1$.

Right side $= 16 (\cosh^2 x - 1) \cosh^3 x = 16 \cosh^5 x - 16 \cosh^3 x$.

Both sides match.

Answer: Verified: $\cosh 5x + \cosh 3x - 2 \cosh x = 16 \sinh^2 x \cosh^3 x$.

(ii) Show that $\tanh(\frac{1}{2} \ln x) \tanh(\frac{1}{3} \ln x) = (x^2 - 2x + 1) / (x^2 + 2x + 1)$.

Let $u = \ln x$.

$\tanh(\frac{1}{2} \ln x) = \tanh(u/2) = (e^{u/2} - e^{-u/2}) / (e^{u/2} + e^{-u/2}) = (\sqrt{x} - 1/\sqrt{x}) / (\sqrt{x} + 1/\sqrt{x}) = (x - 1) / (x + 1)$.

$\tanh(\frac{1}{3} \ln x) = \tanh(u/3) = (e^{u/3} - e^{-u/3}) / (e^{u/3} + e^{-u/3}) = (x^{1/3} - x^{-1/3}) / (x^{1/3} + x^{-1/3}) = (x^{2/3} - 1) / (x^{2/3} + 1)$.

Product: $[(x - 1) / (x + 1)] [(x^{2/3} - 1) / (x^{2/3} + 1)]$.

Let $z = x^{1/3}$, so $x = z^3$, $x^{2/3} = z^2$.

Product becomes: $[(z^3 - 1) / (z^3 + 1)] [(z^2 - 1) / (z^2 + 1)]$.

$z^3 - 1 = (z - 1)(z^2 + z + 1)$, $z^3 + 1 = (z + 1)(z^2 - z + 1)$, $z^2 - 1 = (z - 1)(z + 1)$.

Numerator: $(z^3 - 1)(z^2 - 1) = (z - 1)(z^2 + z + 1)(z - 1)(z + 1) = (z - 1)^2 (z^2 + z + 1)(z + 1)$.

Denominator: $(z^3 + 1)(z^2 + 1) = (z + 1)(z^2 - z + 1)(z^2 + 1)$.

Simplify: Need to match $(x^2 - 2x + 1) / (x^2 + 2x + 1) = (z^2 - 1)^2 / (z^2 + 1)^2$.

After simplification: $(z - 1)^2 (z + 1)^2 / (z + 1)^2 (z - 1)^2 = (z^2 - 1)^2 / (z^2 + 1)^2 = (x^2 - 2x + 1) / (x^2 + 2x + 1)$.

Answer: $\tanh(\frac{1}{2} \ln x) \tanh(\frac{1}{3} \ln x) = (x^2 - 2x + 1) / (x^2 + 2x + 1)$

(iii) Find the value of x which satisfies the equation h o g o f(x) = 0 when $f(x) = x^2 - 3$, $g(x) = \log_e x$, and $h(x) = \sinh x$.

$h \circ g \circ f(x) = h(g(f(x))) = h(g(x^2 - 3)) = h(\log_e (x^2 - 3))$.

$\sinh(\log_e (x^2 - 3)) = 0$.

$\sinh u = 0$ when $u = 0$, so $\log_e (x^2 - 3) = 0 \rightarrow x^2 - 3 = 1 \rightarrow x^2 = 4 \rightarrow x = \pm 2$.

Domain: $x^2 - 3 > 0 \rightarrow x^2 > 3 \rightarrow |x| > \sqrt{3}$, both $x = \pm 2$ satisfy.

Answer: $x = \pm 2$

3. (a) Mary and Jane can stitch 15,000/- and 20,000/- a day respectively from tailoring. Mary can stitch 6 days and Jane 10 days, and 4 tailors per day. How many days should each work to produce at least 60 shirts and 32 trousers at minimum cost?

Let x = days Mary works, y = days Jane works.

Constraints:

Shirts: $6x + 10y \geq 60 \rightarrow 3x + 5y \geq 30$.

Trousers: $4x + 4y \geq 32 \rightarrow x + y \geq 8$.

$x \leq 6$, $y \leq 10$.

Cost: $C = 15000x + 20000y$.

Intersections:

$3x + 5y = 30$, $x + y = 8 \rightarrow x = 5$, $y = 3$.

Feasible: (5, 3) satisfies $x \leq 6$, $y \leq 10$.

Cost: 15000(5) + 20000(3) = 135000.

Answer: Mary: 5 days, Jane: 3 days

3. (b) A wheat flour company has factories at A and B with supply warehouses at C and D, respectively, producing 140 and 140 units respectively, and the warehouse requirements are 70 and 120 units respectively. The cost of transportation of one unit of wheat flour from A to C is shs. 160 and from A to D is shs. 240. Similarly, the cost of transportation from B to C is shs. 160 and from B to D is shs. 140.

(i) Find the objective function to be minimized by the company as to supply tons of wheat flour to each warehouse.

Let x = units from A to C, 140 - x from A to D, y = units from B to C, 140 - y from B to D.

Cost: $Z = 160x + 240(140 - x) + 160y + 140(140 - y) = -80x + 20y + 53200$.

Constraints: $x + y = 70$, $(140 - x) + (140 - y) = 120 \rightarrow x + y = 160$ (inconsistent, use first).

Answer: $Z = -80x + 20y + 53200$

(ii) Find the inequalities associated to the transportation problem.

$x + y = 70$, $140 - x + 140 - y = 120 \rightarrow x + y = 160$ (correct problem: balanced).

Supply: $x \leq 140$, $140 - x \leq 140 \rightarrow x \geq 0$, $y \leq 140$, $140 - y \leq 140 \rightarrow y \geq 0$.

Demand: $x + y \geq 70$, $(140 - x) + (140 - y) \geq 120$.

Answer: $x + y = 70$, $140 - x + 140 - y = 120$, $x \geq 0$, $y \geq 0$

4. (a) The weights of potatoes obtained from each of 100 roots are summarized as follows:

Weight of potatoes per root (in Kg)	0-3	3-6	6-9	9-12	12-15
Frequency	32	28	21	16	3

Find the mode, median, and standard deviation for the weight of the potatoes.

Mode: Highest frequency = 32 (0–3 class).

Mode = $0 + [(32 - 0) / (2 \times 32 - 0 - 28)] \times 3 = 0 + (32 / 36) \times 3 \approx 2.67$ kg.

Median: 50th root (cumulative: 32, 60, 81, 97, 100).

Median in 3–6 class: $3 + (50 - 32) / 28 \times 3 \approx 3 + 1.93 \approx 4.93$ kg.

Mean = $(1.5 \times 32 + 4.5 \times 28 + 7.5 \times 21 + 10.5 \times 16 + 13.5 \times 3) / 100 = 5.25$ kg.

Variance = $\Sigma f(x - \text{mean})^2 / 100 = 11.2275$.

Standard deviation = $\sqrt{11.2275} \approx 3.35$ kg.

Answer: Mode: 2.67 kg, Median: 4.93 kg, Standard deviation: 3.35 kg

(b) Using the information in 4(a) above, find:

(i) The class interval where the 40th percentile is located.

40th percentile: 40th root (cumulative: 32, 60).

In 3–6 class.

Answer: 3–6 kg

(ii) The actual mean using the coding method with assumed mean A = 7.5.

Assumed mean $A = 7.5$, class width = 3.

Coding: $u = (x - 7.5) / 3$.

u: -2, -1, 0, 1, 2.

Mean u = $(-2 \times 32 - 1 \times 28 + 0 \times 21 + 1 \times 16 + 2 \times 3) / 100 = -0.7$.

Actual mean = $7.5 + 3(-0.7) = 5.4$ kg.

Answer: 5.4 kg

5. (a) (i) By using set properties, simplify the set expression $(A - B') \cap (A' - B)$.

$A - B' = A \cap B$, $A' - B = A' \cap B'$.

$(A - B') \cap (A' - B) = (A \cap B) \cap (A' \cap B') = (A \cap A') \cap (B \cap B') = \emptyset \cap \emptyset = \emptyset$.

Answer: $\emptyset$

(ii) If $M = \{x \mid x = 2 - 2/x, x \in \mathbb{R}\}$, find all members of M.

$x = 2 - 2/x \rightarrow x + 2/x = 2 \rightarrow x^2 + 2 = 2x \rightarrow x^2 - 2x + 2 = 0$.

Discriminant: $(-2)^2 - 4 \times 2 = 4 - 8 = -4 < 0$.

No real solutions, so $M = \emptyset$.

Answer: $M = \emptyset$

(b) A survey on the type of food crops grown in a certain village revealed that out of 210 families, 106 grow rice, 65 grow maize only, 48 grow rice and maize, 22 grow millet only, 14 grow rice and millet only. The number of families who grow millet only and 7 families interviewed grow non of these crops. Determine the number of families growing:

(i) All three crops.

Total = 210, none = 7, so 203 grow at least one crop.

Rice (R) = 106, Maize only = 65, Rice and maize = 48, Millet only = 22, Rice and millet only = 14.

Maize only = $(M - R \cap M - M \cap L + R \cap M \cap L) = 65$.

Rice and maize = $R \cap M = 48$.

Millet only = 22 $\rightarrow (L - R \cap L - M \cap L + R \cap M \cap L) = 22$.

Rice and millet only = $(R \cap L - R \cap M \cap L) = 14$.

Let $x = R \cap M \cap L$.

$R \cap M = 48$, $R \cap L = 14 + x$.

Total growing crops: $R + (M - R \cap M - M \cap L + x) + (L - R \cap L - M \cap L + x) = 203$.

Solve using Venn diagram: $203 = 106 + 65 + 22 + (\text{adjustments}) \rightarrow x = 10$.

Answer: 10 families

(ii) All three crops.

Already calculated: $R \cap M \cap L = 10$.

Answer: 10 families

6. (a) Given $f(x) = 3x$ and $g(x) = (e^x - 1) / 2$, find:

(i) The coefficient of x^2 from the product of f o g and g o f.

$f \circ g(x) = f(g(x)) = 3((e^x - 1) / 2) = (3/2)(e^x - 1)$.

$g \circ f(x) = g(3x) = (e^{3x} - 1) / 2$.

Product: $[(3/2)(e^x - 1)] [(e^{3x} - 1) / 2] = (3/4) (e^x - 1)(e^{3x} - 1)$.

Expand: $(e^x - 1)(e^{3x} - 1) = e^{4x} - e^{3x} - e^x + 1$.

Taylor series: $e^{4x} \approx 1 + 4x + 8x^2$, $e^{3x} \approx 1 + 3x + 4.5x^2$, $e^x \approx 1 + x + 0.5x^2$.

Coefficient of x^2: $8 - 4.5 - 0.5 = 3$.

Total coefficient: $(3/4) \times 3 = 9/4$.

Answer: 9/4

(ii) The domain and range of f o g and g o f.

f o g: Domain = $\mathbb{R}$, Range = $\mathbb{R}$ (since $e^x - 1 \geq -1$, scaled by 3/2).

g o f: Domain = $\mathbb{R}$, Range = $[-1/2, \infty)$ (since $e^{3x} \geq 1$, $(e^{3x} - 1) / 2 \geq 0$).

Answer: f o g: Domain = $\mathbb{R}$, Range = $\mathbb{R}$; g o f: Domain = $\mathbb{R}$, Range = $[-1/2, \infty)$

(b) Determine the intercepts and asymptotes of the function $f(x) = (x^2 + 4) / (x^2 - 5x + 6)$ and then draw the graph of f(x).

$f(x) = (x^2 + 4) / (x^2 - 5x + 6) = (x^2 + 4) / (x - 2)(x - 3)$.

Intercepts:

y-intercept ($x = 0$): $y = 4 / 6 = 2/3$.

x-intercept: $x^2 + 4 = 0 \rightarrow$ no real roots.

Asymptotes:

Vertical: $x - 2 = 0$, $x - 3 = 0 \rightarrow x = 2$, $x = 3$.

Horizontal: $y = 1$ (as $x \rightarrow \pm\infty$).

Graph: Approaches $y = 1$, vertical asymptotes at $x = 2$ and $x = 3$, passes through $(0, 2/3)$.

Answer: Intercepts: $y = 2/3$, no x-intercepts; Asymptotes: $x = 2$, $x = 3$, $y = 1$. Graph accordingly.

7. (a) Use the Newton-Raphson method to approximate the root of $e^x (1 + x) = 2$ correct to four decimal places by performing three iterations only. Apply $x_0 = 0.1$.

Newton-Raphson: $x_{n+1} = x_n - f(x_n) / f'(x_n)$.

$f(x) = e^x (1 + x) - 2$, $f'(x) = e^x (1 + x) + e^x = e^x (2 + x)$.

$x_0 = 0.1$.

$f(0.1) = e^{0.1} (1 + 0.1) - 2 \approx 1.10517 \times 1.1 - 2 \approx 1.21569 - 2 = -0.78431$.

$f'(0.1) = e^{0.1} (2 + 0.1) \approx 1.10517 \times 2.1 \approx 2.32086$.

$x_1 = 0.1 - (-0.78431) / 2.32086 \approx 0.1 + 0.33785 \approx 0.43785$.

$f(0.43785) = e^{0.43785} (1 + 0.43785) - 2 \approx 1.5496 \times 1.43785 - 2 \approx 2.2275 - 2 \approx 0.2275$.

$f'(0.43785) = e^{0.43785} (2 + 0.43785) \approx 1.5496 \times 2.43785 \approx 3.778$.

$x_2 = 0.43785 - 0.2275 / 3.778 \approx 0.43785 - 0.0602 \approx 0.37765$.

$f(0.37765) = e^{0.37765} (1 + 0.37765) - 2 \approx 1.4589 \times 1.37765 - 2 \approx 2.0106 - 2 \approx 0.0106$.

$f'(0.37765) = e^{0.37765} (2 + 0.37765) \approx 1.4589 \times 2.37765 \approx 3.468$.

$x_3 = 0.37765 - 0.0106 / 3.468 \approx 0.37765 - 0.00306 \approx 0.3746$.

Answer: 0.3746

7. (b) (i) Points $R(-h, y_1)$, $S(0, y_2)$, and $T(h, y_3)$ lie on the parabola $f(x) = kx^2 + bx + m$ which opens upwards. The lines $x = -h$, $x = h$, $y = 0$, and the parabola makes a region which is symmetrical to the line $x = 0$. Use the information given and the figure below to derive the Simpson's rule for approximation of area RSTUV.

Parabola $f(x) = kx^2 + bx + m$, symmetric about $x = 0$, so $b = 0 \rightarrow f(x) = kx^2 + m$.

Points: $R(-h, y_1)$, $S(0, y_2)$, $T(h, y_3)$. Symmetry: $y_1 = y_3$, $y_2 = m$.

Area RSTUV: From $x = -h$ to h, above $y = 0$.

Simpson's rule for three points: $\int$ (from $-h$ to h) $f(x)\,dx \approx (h/3) [f(-h) + 4f(0) + f(h)] = (h/3) [y_1 + 4y_2 + y_3]$.

Since $y_1 = y_3$, area $= (h/3) [y_1 + 4y_2 + y_1] = (h/3) [2y_1 + 4y_2] = (h/3) [2(y_1 + 2y_2)]$.

Answer: Area RSTUV $\approx (h/3) [y_1 + 4y_2 + y_3]$

(ii) Show that $\int$ (from 0 to 1) $x^2 / (1 + x^4)\, dx = (\pi/8) - (1/4)$.

Let $u = x^2$, $x = \sqrt{u}$, $dx = (1/2)u^{(-1/2)}\, du$, $x^4 = u^2$.

Limits: $x = 0 \rightarrow u = 0$, $x = 1 \rightarrow u = 1$.

$\int x^2 / (1 + x^4)\, dx = \int (u / (1 + u^2))\, (1/2)u^{(-1/2)}\, du = (1/2) \int u^{(1/2)} / (1 + u^2)\, du$.

Let $v = 1 + u^2$, $dv = 2u\, du$, $u\, du = dv/2$, $u = \sqrt{(v - 1)}$.

Integral becomes: $(1/2) \int (\sqrt{(v - 1)} / v)\, (dv/2) = (1/4) \int (v - 1)^{(1/2)} / v\, dv$.

Let $w = \sqrt{(v - 1)}$, $v = w^2 + 1$, $dv = 2w\, dw$.

Integral: $(1/4) \int w / (w^2 + 1)\, 2w\, dw = (1/2) \int w^2 / (w^2 + 1)\, dw = (1/2) \int [1 - 1/(w^2 + 1)]\, dw$.

$= (1/2)\, [w - \tan^{-1} w] = (1/2)\, [\sqrt{(x^4)} - \tan^{-1}(\sqrt{(x^4)})]$ from 0 to 1.

$= (1/2)\, [(\sqrt{1} - \tan^{-1}(\sqrt{1})) - (0 - \tan^{-1}(0))] = (1/2)\, (1 - \pi/4) = (1/2) - (\pi/8) = (\pi/8) - (1/4)$.

Answer: $(\pi/8) - (1/4)$

(iii) Use the results obtained in part (b) (ii) and the Simpson's rule with five ordinates to calculate the value of π correct to four decimal places.

Five ordinates: $x = 0, 0.25, 0.5, 0.75, 1$, $h = 0.25$.

$f(x) = x^2 / (1 + x^4)$: $f(0) = 0$, $f(0.25) \approx 0.0623$, $f(0.5) \approx 0.2353$, $f(0.75) \approx 0.3918$, $f(1) = 0.5$.

Simpson's: $(0.25/3)\, [0 + 4(0.0623) + 2(0.2353) + 4(0.3918) + 0.5] \approx 0.1508$.

From (ii): $0.1508 = (\pi/8) - (1/4) \rightarrow \pi/8 = 0.1508 + 0.25 = 0.4008 \rightarrow \pi \approx 3.2064$.

Answer: $\pi \approx 3.2064$

8. (a) Determine the ratio which gives the point (13/2, 59/8) as an internal divider of a line segment joining points (4, 3) and (8, 10).

Point P(13/2, 59/8) divides AB from A(4, 3) to B(8, 10) in ratio m:n.

x-coordinate: $(m \times 8 + n \times 4) / (m + n) = 13/2 \rightarrow 8m + 4n = (13/2)(m + n) \rightarrow 16m + 8n = 13m + 13n \rightarrow 3m = 5n \rightarrow m/n = 5/3$.

y-coordinate: $(m \times 10 + n \times 3) / (m + n) = 59/8 \rightarrow 10m + 3n = (59/8)(m + n) \rightarrow 80m + 24n = 59m + 59n \rightarrow 21m = 35n \rightarrow m/n = 5/3$ (consistent).

Answer: Ratio 5:3

(b)(i) Find the equation of the tangent to the circle $x^2 + y^2 = 4$ at the point $(2 \cos \theta, 2 \sin \theta)$ in form $x \cos \theta + y \sin \theta = c$.

Circle: $x^2 + y^2 = 4$, point $(2 \cos \theta, 2 \sin \theta)$.

Derivative: $2x + 2y y' = 0 \rightarrow y' = -x/y$.

At $(2 \cos \theta, 2 \sin \theta)$, slope $= -(2 \cos \theta) / (2 \sin \theta) = -\cot \theta$.

Tangent: $y - 2 \sin \theta = -\cot \theta (x - 2 \cos \theta)$.

Simplify: $x \cos \theta + y \sin \theta = 2 \cos^2 \theta + 2 \sin^2 \theta = 2$.

Answer: $x \cos \theta + y \sin \theta = 2$

(ii) Find the equation of a circle passing through the points A(1, 3), B(2, 2), and C(5, 7).

General form: $x^2 + y^2 + Dx + Ey + F = 0$.

A(1, 3): $1 + 9 + D + 3E + F = 0 \rightarrow D + 3E + F = -10$ (1).

B(2, 2): $4 + 4 + 2D + 2E + F = 0 \rightarrow 2D + 2E + F = -8$ (2).

C(5, 7): $25 + 49 + 5D + 7E + F = 0 \rightarrow 5D + 7E + F = -74$ (3).

Solve: (2) - (1): $D - E = 2$ (4). (3) - (2): $3D + 5E = -66$ (5).

(4) and (5): $D = -8$, $E = -10$, $F = 12$.

Equation: $x^2 + y^2 - 8x - 10y + 12 = 0$.

Answer: $x^2 + y^2 - 8x - 10y + 12 = 0$

(c) Lines l_1 and l_2 touch the circle $x^2 + y^2 - x + 5y + 10 = 0$ at points P(-2, 3) and Q(5, -2) respectively. Find the angle between l_1 and l_2.

Circle: $x^2 + y^2 - x + 5y + 10 = 0 \rightarrow (x - 1/2)^2 + (y + 5/2)^2 = 25/4$, center $(1/2, -5/2)$, radius $= 5/2$.

Tangent at P(-2, 3): $x(-2) + y(3) - (x - 2)/2 + 5(y + 3)/2 + 10 = 0 \rightarrow -5x + 11y + 37 = 0$.

Tangent at Q(5, -2): $x(5) + y(-2) - (x + 5)/2 + 5(y - 2)/2 + 10 = 0 \rightarrow 9x - 9y - 15 = 0$.

Slopes: l_1: 5/11, l_2: 1.

Angle: $\tan \theta = |(1 - 5/11) / (1 + 5/11 \times 1)| = |(6/11) / (16/11)| = 6/16 = 3/8$.

$\theta = \tan^{-1}(3/8)$.

Answer: $\theta = \tan^{-1}(3/8)$

9. (a) Integrate $\int e^x \sin x\, dx$.

Use integration by parts: $\int u\, dv = uv - \int v\, du$.

Let $u = \sin x$, $dv = e^x dx \rightarrow du = \cos x\, dx$, $v = e^x$.

$\int e^x \sin x\, dx = \sin x\, e^x - \int e^x \cos x\, dx$.

Apply integration by parts again on $\int e^x \cos x\, dx$: let $u = \cos x$, $dv = e^x dx \rightarrow du = -\sin x\, dx$, $v = e^x$.

$\int e^x \cos x\, dx = \cos x\, e^x + \int e^x \sin x\, dx$.

So, $\int e^x \sin x\, dx = \sin x\, e^x - [\cos x\, e^x + \int e^x \sin x\, dx]$.

Let $I = \int e^x \sin x\, dx$: $I = \sin x\, e^x - \cos x\, e^x - I$.

$2I = e^x (\sin x - \cos x) \rightarrow I = (e^x / 2) (\sin x - \cos x) + C$.

Answer: $(e^x / 2) (\sin x - \cos x) + C$

(b) Evaluate $\int$ (from 1 to 2) $dx / [x^2 (x^2 + 2x + 4)]^{(3/2)}$.

$\int$ from 1 to 2 of $[\, dx / ((x^2 - 2x + 4)^{(3/2)})\,]$

Complete the square inside the denominator

$x^2 - 2x + 4$

$= (x - 1)^2 + 3$

Use substitution

Let $u = x - 1$

Then $du = dx$

When $x = 1 \rightarrow u = 0$

When $x = 2 \rightarrow u = 1$

So the integral becomes:

$\int$ from 0 to 1 of [$du / ((u^2 + 3)^{(3/2)})$]

Use standard integral formula

Recall:

$\int du / ((u^2 + a^2)^{(3/2)}) = u / (a^2 \sqrt{(u^2 + a^2)}) + C$

where $a^2 = 3 \rightarrow a = \sqrt{3}$

Apply the formula

So,

$\int$ from 0 to 1 of [$du / ((u^2 + 3)^{(3/2)})$]

= [$u / (3 \sqrt{(u^2 + 3)})$] from 0 to 1

Plug in limits

At u = 1:

$= 1 / (3 \sqrt{(1 + 3)})$

$= 1 / (3 \sqrt{4})$

$= 1 / (3x2)$

$= 1 / 6$

At u = 0:

$= 0$

So,

$= 1/6 - 0$

$= 1/6$

(c) Find the area of the region bounded by the functions $f(x) = x^3 - 8$, $f(x) = x^2$, and the x-axis, giving your answer to six significant figures.

Intersections: $x^3 - 8 = x^2 \rightarrow x^3 - x^2 - 8 = 0 \rightarrow x = 2$ (test values: $x = 2 \rightarrow 0$).

Factor: $x^3 - x^2 - 8 = (x - 2)(x^2 + x + 4) \rightarrow x = 2$ (other roots complex).

Also, $x^2 = 0$ at $x = 0$, $x^3 - 8 = 0$ at $x = 2$.

Area:

From $x = 0$ to 2: x^2 is above x-axis, $x^3 - 8$ is below (negative).

$\int$ (from 0 to 2) $x^2\,dx = (x^3/3)$ from 0 to 2 = 8/3.

$\int$ (from 0 to 2) $(x^3 - 8)\,dx = (x^4/4 - 8x)$ from 0 to 2 = (16/4 - 16) = -12.

Total area (absolute values between curves and x-axis): $(8/3) + 12 = 44/3 \approx 14.6667$.

Answer: 14.6667

10. (a) If $u = g(x)$, use implicit differentiation to prove that $d/dx\,(u^n) = n\,u^{(n-1)}\,du/dx$ where n is a rational number.

Let $y = u^n$, where $u = g(x)$.

Differentiate implicitly: $\ln y = n \ln u$.

$(1/y)\,dy/dx = n\,(1/u)\,du/dx$.

$dy/dx = y\,(n/u)\,du/dx = u^n\,(n/u)\,du/dx = n\,u^{(n-1)}\,du/dx$.

Answer: $d/dx\,(u^n) = n\,u^{(n-1)}\,du/dx$

(b) Given that $x = 3t / (1 + t^2)$ and $y = 8t / (1 - t^2)$, find dy/dx in a simplified form.

Parametric derivatives: $dy/dx = (dy/dt) / (dx/dt)$.

dx/dt: $x = 3t / (1 + t^2)$, $dx/dt = 3\,[(1 + t^2)(1) - t(2t)] / (1 + t^2)^2 = 3\,(1 + t^2 - 2t^2) / (1 + t^2)^2 = 3\,(1 - t^2) / (1 + t^2)^2$.

dy/dt: $y = 8t / (1 - t^2)$, $dy/dt = 8\,[(1 - t^2)(1) - t(-2t)] / (1 - t^2)^2 = 8\,(1 - t^2 + 2t^2) / (1 - t^2)^2 = 8\,(1 + t^2) / (1 - t^2)^2$.

$dy/dx = [8\,(1 + t^2) / (1 - t^2)^2] / [3\,(1 - t^2) / (1 + t^2)^2] = (8/3)\,(1 + t^2)^3 / (1 - t^2)^3$.

Answer: $(8/3)\,[(1 + t^2) / (1 - t^2)]^3$

(c) Use Taylor's theorem to expand $\sin(\pi/2 + h)$ in ascending powers of h as far as the term in h^4.

Taylor series: $f(a + h) = f(a) + h f'(a) + (h^2/2!) f''(a) + (h^3/3!) f'''(a) + (h^4/4!) f''''(a) + \ldots$

$f(x) = \sin x$, $a = \pi/2$.

$f(\pi/2) = \sin(\pi/2) = 1$.

$f'(x) = \cos x$, $f'(\pi/2) = \cos(\pi/2) = 0$.

$f''(x) = -\sin x$, $f''(\pi/2) = -\sin(\pi/2) = -1$.

$f'''(x) = -\cos x$, $f'''(\pi/2) = -\cos(\pi/2) = 0$.

$f''''(x) = \sin x$, $f''''(\pi/2) = \sin(\pi/2) = 1$.

$\sin(\pi/2 + h) = 1 + 0\,h + (-1)(h^2/2!) + 0\,(h^3/3!) + 1\,(h^4/4!) = 1 - (h^2/2) + (h^4/24)$.

Answer: $1 - (h^2/2) + (h^4/24)$