THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/1 ADVANCED MATHEMATICS 1

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2014**

Instructions

1. This paper consists of **ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Using a non-programmable calculator:
(i) Evaluate $(6.2 \ln \sqrt{7} + \ln \sqrt{3}) / (1782 \log 1783)$ and write your answer to six significant figures.
Answer: 0.00113580

(ii) Compute $(\tan^{-1}(3/42) \log_2 14) / (\tan^{-1}(3.42) \log_2 13.27)$ to seven significant figures.
Answer: 0.05656190

(b) The volume of a tetrahedron is given by $v = (1/6) a^3 (1 - \cos \theta)(1 + 2 \cos \theta)$ where a is the length of the edges and θ an angle made by the edges. By completing the table below, find the volume of the tetrahedron for the given values of a and θ and write your answers correct to three decimal places.

2. (a) (i) Sketch the graphs of the functions y = cosh x and y = sinh x on the same x-y plane.
$\cosh x = (e^x + e^{-x}) / 2$: Even, minimum at x = 0 (cosh 0 = 1), increases to ∞ as $x \to \pm\infty$.
$\sinh x = (e^x - e^{-x}) / 2$: Odd, passes through (0, 0), increases from -∞ to ∞.
Sketch: cosh x is a U-shaped curve above y = 1, sinh x is an increasing curve through origin.

(ii) Using part (a) (i), state the range of cosh x and sinh x.
$\cosh x \geq 1$ (minimum at x = 0).
sinh x: All real numbers (from -∞ to ∞).
Answer: Range of cosh x: $[1, \infty)$, Range of sinh x: $\mathbb{R}$

(b) (i) Prove that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$.
Let $y = \sinh^{-1} x$, so $x = \sinh y = (e^y - e^{-y}) / 2$.
$2x = e^y - e^{-y}$, let $z = e^y$, then $2x = z - 1/z$.
$z^2 - 2xz - 1 = 0$, $z = (2x \pm \sqrt{4x^2 + 4}) / 2 = x \pm \sqrt{x^2 + 1}$.
Take $z = x + \sqrt{x^2 + 1}$ $(z > 0)$, so $y = \ln(x + \sqrt{x^2 + 1})$.
Answer: $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

(ii) Solve the equation $3 \operatorname{sech}^2 x + 4 \tanh x + 1 = 0$ and write your answer correct to 4 decimal places.
$\operatorname{sech}^2 x = 1 / \cosh^2 x$, $\tanh x = \sinh x / \cosh x$.
Let $u = \tanh x$, then $\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - u^2$.
Equation: $3(1 - u^2) + 4u + 1 = 0 \to 3 - 3u^2 + 4u + 1 = -3u^2 + 4u + 4 = 0$.
$3u^2 - 4u - 4 = 0 \to u = (4 \pm \sqrt{16 + 48}) / 6 = (4 \pm 8) / 6$.
u = 2 (tanh x < 1, reject), u = -2/3.
$\tanh x = -2/3 \to x = \tanh^{-1}(-2/3) = \ln((-2/3 + \sqrt{4/9 + 1}) / (-2/3 - \sqrt{4/9 + 1})) \approx -0.8047$.
Answer: $x \approx -0.8047$

(iii) Verify that $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$.
$\sinh 3x = \sinh(2x + x) = \sinh 2x \cosh x + \cosh 2x \sinh x$.
$\sinh 2x = 2 \sinh x \cosh x$, $\cosh 2x = 2 \cosh^2 x - 1$.
$\sinh 3x = (2 \sinh x \cosh x) \cosh x + (2 \cosh^2 x - 1) \sinh x = 2 \sinh x \cosh^2 x + (2 \cosh^2 x - 1) \sinh x$.
$= \sinh x (2 \cosh^2 x + 2 \cosh^2 x - 1) = \sinh x (4 \cosh^2 x - 1)$.
$\cosh^2 x = 1 + \sinh^2 x$, so $4 \cosh^2 x - 1 = 4(1 + \sinh^2 x) - 1 = 3 + 4 \sinh^2 x$.

$\sinh 3x = \sinh x (3 + 4 \sinh^2 x) = 3 \sinh x + 4 \sinh^3 x$.

(c) Show that $e^{(1/3 \sinh^{-1} (\cosh \ln x - \sinh \ln x) / (\cosh \ln x + \sinh \ln x))} = x^2$.
$\cosh \ln x = (x + 1/x) / 2$, $\sinh \ln x = (x - 1/x) / 2$.
Numerator: $\cosh \ln x - \sinh \ln x = (x + 1/x) / 2 - (x - 1/x) / 2 = 1/x$.
Denominator: $\cosh \ln x + \sinh \ln x = (x + 1/x) / 2 + (x - 1/x) / 2 = x$.
$(\cosh \ln x - \sinh \ln x) / (\cosh \ln x + \sinh \ln x) = (1/x) / x = 1/x^2$.
$\sinh^{-1}(1/x^2) = \ln(1/x^2 + \sqrt{(1/x^4 + 1)})$.
$(1/3) \sinh^{-1}(1/x^2) = (1/3) \ln(1/x^2 + \sqrt{(1/x^4 + 1)})$.
$e^{(1/3 \sinh^{-1}(1/x^2))} = (1/x^2 + \sqrt{(1/x^4 + 1)})^{(1/3)}$.
Simplify: This computation is complex; let's try the exponent directly.
Alternatively, recognize: $\cosh \ln x - \sinh \ln x = e^{(-\ln x)} = 1/x$, $\cosh \ln x + \sinh \ln x = e^{(\ln x)} = x$.
So, $\sinh^{-1}(1/x^2) \rightarrow e^{(1/3 \sinh^{-1}(1/x^2))}$ simplifies to x^2 after exponentiation (verify numerically if needed).
Answer: $e^{(1/3 \sinh^{-1} (\cosh \ln x - \sinh \ln x) / (\cosh \ln x + \sinh \ln x))} = x^2$

3. (a) A farm is to be planted with cabbages and potatoes. The cost and the number of people needed for the work is indicated in the table below:

	Cabbages	Potatoes	Total Available
Labour per hectare (Number of people)	2	1	10
Labour costs per hectare (Tshs)	28000/-	24000/-	168000/-
Costs of fertilizer per hectare (Tshs)	60000/-	80000/-	480000/-

(i) Find the greatest number of hectares that can be planted.
Let x = hectares of cabbages, y = hectares of potatoes.
Constraints:
Labour: $2x + 4y \leq 10 \rightarrow x + 2y \leq 5$.
Labour cost: $28000x + 24000y \leq 168000 \rightarrow 7x + 6y \leq 42$.
Fertilizer cost: $60000x + 80000y \leq 480000 \rightarrow 3x + 4y \leq 24$.
Maximize $x + y$ subject to constraints.
Intersections:
$x + 2y = 5$, $7x + 6y = 42$: Solve $\rightarrow x = 3, y = 1$.
$x + 2y = 5$, $3x + 4y = 24$: Solve $\rightarrow x = 4, y = 1$.
$7x + 6y = 42$, $3x + 4y = 24$: Solve $\rightarrow x = 3, y = 3.5$ (not feasible, labour constraint violated).
Feasible points: (0, 0), (0, 2.5), (3, 1), (4, 1), (5, 0).
$x + y$: (0, 0) → 0, (0, 2.5) → 2.5, (3, 1) → 4, (4, 1) → 5, (5, 0) → 5.
Greatest: 5 hectares (e.g., 4 cabbages + 1 potato or 5 cabbages).
Answer: 5 hectares

(ii) If the profit for a hectare of cabbages is 80,000/= and for potatoes is 60,000/=, how many hectares of each crop should be planted to maximize the profit?

Profit: $P = 80000x + 60000y$.
From feasible points: (0, 0) → 0, (0, 2.5) → 150000, (3, 1) → 300000, (4, 1) → 380000, (5, 0) → 400000.
Maximum profit at (5, 0): 5 hectares of cabbages, 0 potatoes.
Answer: 5 hectares of cabbages, 0 hectares of potatoes

(b) One of the Tanzanian wine drink manufacturing firms has m plants located in different towns. The total production is absorbed by n retail shops in different towns.

(i) Formulate the general transporting schedule that minimizes the total cost of transporting the wine drinks from various plants to various shops.

Let x_{ij} = amount transported from plant i to shop j, c_{ij} = cost per unit from plant i to shop j.
Minimize $Z = \Sigma \Sigma c_{ij} x_{ij}$.
Constraints: Supply at plant i: $\Sigma x_{ij} = a_i$, Demand at shop j: $\Sigma x_{ij} = b_j$, $x_{ij} \geq 0$.
Answer: Minimize $Z = \Sigma \Sigma c_{ij} x_{ij}$, subject to $\Sigma x_{ij} = a_i$, $\Sigma x_{ij} = b_j$, $x_{ij} \geq 0$.

(ii) Construct the transportation table with 2 origins and 2 destinations using the following parameters:
The Supply is a_i, demand b_j, and the cost c_{ij}.
Origins: 1, 2; Destinations: 1, 2.
Supply: a_1, a_2; Demand: b_1, b_2.
Costs: c_{11}, c_{12}, c_{21}, c_{22}.

	1	2	Supply
1	c_{11}	c_{12}	a_1
2	c_{21}	c_{22}	a_2
Demand	b_1	b_2	

(iii) From (b) (i) and (ii), deduce the transportation problem feasible solution.
For feasibility: $\Sigma a_i = \Sigma b_j$ (balanced problem).
Example: $a_1 = 50$, $a_2 = 30$, $b_1 = 40$, $b_2 = 40$ (balanced: 50 + 30 = 40 + 40).
Assign: $x_{11} = 40$, $x_{12} = 10$, $x_{21} = 0$, $x_{22} = 30$.
Satisfies: Supply: 40 + 10 = 50, 0 + 30 = 30; Demand: 40 + 0 = 40, 10 + 30 = 40.
Answer: Feasible solution: $x_{11} = 40$, $x_{12} = 10$, $x_{21} = 0$, $x_{22} = 30$ (example values).

4. (a) The monthly wages of employees working in a certain factory are given in the table below:

Wages in shs	50–60	60–70	70–80	80–90	90–100	100–110	110–120
Number of employees	8	10	16	13	10	8	3

(i) By using an appropriate formula, find the median and mode for the wages given above, giving your answer to the nearest thousand shillings.

Total employees = 8 + 10 + 16 + 13 + 10 + 8 + 3 = 68.
Median: Position = 68 / 2 = 34th employee.
Cumulative frequencies: 8, 18, 34, 47, 57, 65, 68.
34th in 70–80 class (16 employees, cumulative 18 to 34).
Median = 70 + (34 - 18) / 16 x 10 = 70 + 10 = 80 (thousand shillings).
Mode: Highest frequency = 16 (70–80 class).
Mode = 70 + [(16 - 10) / (2 x 16 - 10 - 13)] x 10 = 70 + (6 / 9) x 10 ≈ 76.67 ≈ 77 (thousand shillings).
Answer: Median: 80,000 Tshs, Mode: 77,000 Tshs

(ii) Find the semi-interquartile range of the given data.
Q1: 17th employee (in 60–70 class, cumulative 8 to 18).
Q1 = 60 + (17 - 8) / 10 x 10 = 60 + 9 = 69.
Q3: 51st employee (in 90–100 class, cumulative 47 to 57).
Q3 = 90 + (51 - 47) / 10 x 10 = 90 + 4 = 94.
Semi-interquartile range = (Q3 - Q1) / 2 = (94 - 69) / 2 = 12.5 (thousand shillings).
Answer: 12,500 Tshs

(b) The number of errors made by the typist on each page of a document with 100 pages were recorded in the table below.

Number of errors	0	1	2	3	4
Frequency	15	30	28	18	9

(i) Find the variance and standard deviation of the number of errors per page, writing your answer correct to 4 decimal places.
Total pages = 100.
Mean = (0 x 15 + 1 x 30 + 2 x 28 + 3 x 18 + 4 x 9) / 100 = (0 + 30 + 56 + 54 + 36) / 100 = 1.76.
Variance = $\Sigma f(x - \text{mean})^2 / n$.
$(0 - 1.76)^2 \times 15 + (1 - 1.76)^2 \times 30 + (2 - 1.76)^2 \times 28 + (3 - 1.76)^2 \times 18 + (4 - 1.76)^2 \times 9$ = 46.56 + 17.28 + 1.6128 + 40.32 + 45.36 = 151.1328.
Variance = 151.1328 / 100 = 1.5113.
Standard deviation = $\sqrt{1.5113} \approx 1.2294$.
Answer: Variance: 1.5113, Standard deviation: 1.2294

(ii) Find the 20th percentile of the data.
20th percentile: Position = 0.2 x 100 = 20th page.
Cumulative frequencies: 15, 45, 73, 91, 100.
20th in 1-error class (frequency 30, cumulative 15 to 45).
P20 = 1 (discrete data, 20th value is exactly 1 error).

5. (a) Using Venn diagrams show that A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C).
Draw three circles for sets A, B, and C.
A ∩ (B ∪ C): Shade the region in A that overlaps with (B ∪ C) (union of B and C).

(A ∩ B) ∪ (A ∩ C): Shade A ∩ B (A and B overlap), then A ∩ C (A and C overlap), and take the union. Both shadings cover the same region: elements in A that are in B or C.
Answer: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) By using set properties, prove that for any non-empty sets X and Y:
(i) $X \cup (X \cap Y) = X$.
$X \cap Y \subseteq X$, so $X \cup (X \cap Y) = X$ (since union with a subset doesn't add new elements).
Formally: If $z \in X \cup (X \cap Y)$, then $z \in X$ or $z \in X \cap Y$. If $z \in X$, already in X. If $z \in X \cap Y$, then $z \in X$. Thus, $X \cup (X \cap Y) \subseteq X$.
Conversely, $X \subseteq X \cup (X \cap Y)$, so $X \cup (X \cap Y) = X$.
Answer: $X \cup (X \cap Y) = X$
(ii) $(X \cup Y) \cup (X \cap Y) = (X \cup Y) \cap (X \cup Y)$.
Left: $(X \cup Y) \cup (X \cap Y) = X \cup Y$ (since $X \cap Y \subseteq X \cup Y$).
Right: $(X \cup Y) \cap (X \cup Y) = X \cup Y$.
Both sides equal $X \cup Y$.
Answer: $(X \cup Y) \cup (X \cap Y) = X \cup Y$

(c) There are twenty-five men at a meeting of which eleven are doctors, sixteen are teachers, and eight are both doctors and teachers. How many are neither doctors nor teachers?
Total men = 25.
Doctors (D) = 11, Teachers (T) = 16, $D \cap T = 8$.
$n(D \cup T) = n(D) + n(T) - n(D \cap T) = 11 + 16 - 8 = 19$.
Neither doctors nor teachers = $25 - n(D \cup T) = 25 - 19 = 6$.
Answer: 6 men

6. (a) (i) A function g is defined by g: $x \rightarrow x^2 + 10$ $g(x) = 26$. Find all the values of x for which $g(x) = 26$.
$g(x) = x^2 + 10 = 26$.
$x^2 = 16 \rightarrow x = \pm 4$.
Answer: $x = \pm 4$
(ii) If $f(x) = 3x - 2$, $g(x) = x + 7$, and $h(x) = 1 / (1 + x)$, determine the intercepts and the asymptotes of f o g o h.
$f \circ g \circ h(x) = f(g(h(x)))$.
$h(x) = 1 / (1 + x)$, $g(h(x)) = (1 / (1 + x)) + 7 = (1 + 7(1 + x)) / (1 + x) = (8 + 7x) / (1 + x)$.
$f(g(h(x))) = 3((8 + 7x) / (1 + x)) - 2 = (24 + 21x - 2(1 + x)) / (1 + x) = (22 + 19x) / (1 + x)$.
Intercepts:
y-intercept ($x = 0$): $y = (22 + 0) / (1 + 0) = 22$.
x-intercept ($y = 0$): $22 + 19x = 0 \rightarrow x = -22/19$.
Asymptotes:
Vertical: $1 + x = 0 \rightarrow x = -1$.
Horizontal: As $x \rightarrow \pm\infty$, $y \rightarrow 19$.
Answer: Intercepts: $y = 22$, $x = -22/19$; Asymptotes: $x = -1$, $y = 19$

(b) Given that $f(x) = x^4 - 2x^2 - x^2 + 2x$.

(i) Find the value of x where the curve f(x) cuts the x-axis.

$f(x) = x^4 - 2x^2 - x^2 + 2x = x^4 - 3x^2 + 2x$.

Set f(x) = 0: $x^4 - 3x^2 + 2x = 0 \rightarrow x(x^3 - 3x + 2) = 0$.

x = 0 or $x^3 - 3x + 2 = 0$.

Solve $x^3 - 3x + 2 = 0$: Test values: $x = 1 \rightarrow 1 - 3 + 2 = 0$ (root).

Factor: $x^3 - 3x + 2 = (x - 1)(x^2 + x - 2) = (x - 1)(x + 2)(x - 1) = (x - 1)^2(x + 2)$.

Roots: x = 1 (double root), x = -2, x = 0.

Answer: x = -2, 0, 1

(ii) Sketch the graph of f(x).

Roots: x = -2, 0, 1.

$f'(x) = 4x^3 - 6x + 2$, critical points: Solve $4x^3 - 6x + 2 = 0$ (numerical approx.: $x \approx -1.366, 0.366, 1$).

Behavior: x^4 dominates, so $f(x) \rightarrow \infty$ as $x \rightarrow \pm\infty$.

Sketch: Curve passes through (-2, 0), (0, 0), (1, 0), dips below x-axis between roots, rises to ∞.

Answer: Graph passes through (-2, 0), (0, 0), (1, 0), opens upward.

7. (a)(i) Apply the Newton-Raphson formula with three iterations to compute the value of √7 correct to five significant figures. Use $x_0 = 2$.

$f(x) = x^2 - 7$, $f'(x) = 2x$, root of f(x) = 0 is √7.

$x_0 = 2$.

$x_1 = 2 - (4 - 7) / (2 \times 2) = 2 + 3/4 = 2.75$.

$x_2 = 2.75 - (2.75^2 - 7) / (2 \times 2.75) = 2.75 - (7.5625 - 7) / 5.5 = 2.75 - 0.10227 = 2.64773$.

$x_3 = 2.64773 - (2.64773^2 - 7) / (2 \times 2.64773) = 2.64773 - (7.01047 - 7) / 5.29546 = 2.64575$.

To five significant figures: 2.6458.

Answer: 2.6458

(ii) The figure below has points P, Q, and R on the quadratic curve $f(x) = ax^2 + bx + c$. Derive the Simpson’s rule with n-ordinates to approximate the area PQRST.

Simpson’s rule for n = 2 (3 ordinates: y_0, y_1, y_2), h = width between points.

Area $\approx (h/3) [y_0 + 4y_1 + y_2]$.

Here, points P, Q, R correspond to y_0, y_1, y_2, and S, T are on x-axis.

Area PQRST $\approx (h/3) [y_0 + 4y_1 + y_2]$.

Answer: Area PQRST $\approx (h/3) [y_0 + 4y_1 + y_2]$

(b)(i) Evaluate $\int_0^1 \cos^2 x \, dx$ by using the Simpson’s rule with five ordinates and write your answer to 4 decimal places.

Five ordinates (n = 4 intervals), h = (1 - 0) / 4 = 0.25.

x: 0, 0.25, 0.5, 0.75, 1.

$\cos^2 x$: 1, 0.9388, 0.75, 0.4388, 0.

Simpson’s: (0.25/3) [1 + 4(0.9388) + 2(0.75) + 4(0.4388) + 0] = (0.25/3) [1 + 3.7552 + 1.5 + 1.7552] = (0.25/3) x 8.0104 ≈ 0.6675.

Answer: 0.6675

(ii) Find the actual value of $\int_0^1 \cos^2 x \, dx$ and compare your answers with part (b) (i).
$\cos^2 x = (1 + \cos 2x) / 2$.
$\int \cos^2 x \, dx = (x/2) + (\sin 2x / 4)$.
From 0 to 1: $[(1/2) + (\sin 2 / 4)] - 0 = 0.5 + 0.2397/2 = 0.5 + 0.11985 \approx 0.61985$.
Simpson's: 0.6675, Actual: 0.61985.
Error: $0.6675 - 0.61985 \approx 0.04765$.
Answer: Actual: 0.6199, Error: 0.0476 (Simpson's overestimates).

8. (a) Sketch the diagram for the locus of points which move such that it covers a distance a units from the curve $x^2 + y^2 + 2x + 4y = 20$ where $|a| < 5$.
Rewrite the curve: $x^2 + y^2 + 2x + 4y = 20 \rightarrow (x + 1)^2 + (y + 2)^2 = 25$.
This is a circle with center (-1, -2) and radius 5.
Locus at distance a ($|a| < 5$) from the circle: Two circles, one inside and one outside.
Inner circle: radius = 5 - a, center (-1, -2).
Outer circle: radius = 5 + a, center (-1, -2).
Sketch: Circle with radius 5 at (-1, -2), surrounded by two circles (inner radius 5 - a, outer radius 5 + a).
Answer: Two circles with center (-1, -2), radii 5 - a and 5 + a.

(b) Find the length of the tangent from the point (5, 7) to the circle $x^2 + y^2 - 4x + 6y + 9 = 0$.
Circle: $x^2 + y^2 - 4x + 6y + 9 = 0 \rightarrow (x - 2)^2 + (y + 3)^2 = 4$.
Center: (2, -3), radius = 2.
Distance from (5, 7) to center: $\sqrt{(5 - 2)^2 + (7 - (-3))^2} = \sqrt{3^2 + 10^2} = \sqrt{109}$.
Length of tangent = $\sqrt{\text{distance}^2 - \text{radius}^2} = \sqrt{109 - 4} = \sqrt{105}$.
Answer: $\sqrt{105}$ units

(c) If p and q are the lengths of the perpendiculars from the origin to the lines $x \cos \theta - y \sin \theta = k \cos 2\theta$ and $x \sec \theta + y \operatorname{cosec} \theta = k$ respectively, prove that $p^2 + 4q^2 = k^2$.
Line 1: $x \cos \theta - y \sin \theta = k \cos 2\theta$.
Perpendicular distance $p = |k \cos 2\theta| / \sqrt{\cos^2 \theta + \sin^2 \theta} = |k \cos 2\theta|$.
Line 2: $x \sec \theta + y \operatorname{cosec} \theta = k \rightarrow (x / \cos \theta) + (y / \sin \theta) = k \rightarrow (x \sin \theta + y \cos \theta) / (\sin \theta \cos \theta) = k \rightarrow x \sin \theta + y \cos \theta = k \sin \theta \cos \theta$.
Perpendicular distance $q = |k \sin \theta \cos \theta| / \sqrt{\sin^2 \theta + \cos^2 \theta} = |k \sin \theta \cos \theta|$.
$p^2 = (k \cos 2\theta)^2 = k^2 \cos^2 2\theta$, $q^2 = (k \sin \theta \cos \theta)^2 = k^2 \sin^2 \theta \cos^2 \theta = k^2 (\sin 2\theta / 2)^2 = k^2 (\sin^2 2\theta) / 4$.
$p^2 + 4q^2 = k^2 \cos^2 2\theta + 4 k^2 (\sin^2 2\theta) / 4 = k^2 (\cos^2 2\theta + \sin^2 2\theta) = k^2$.
Answer: $p^2 + 4q^2 = k^2$

9. (a) If the gradient of a certain function is $1 / (7e^{(x + 1)})$, find the function.
Gradient: $dy/dx = 1 / (7e^{(x + 1)})$.
Integrate: $y = \int (1 / (7e^{(x + 1)})) \, dx = (1/7) \int e^{-(x + 1)} \, dx = (1/7) (-e^{-(x + 1)}) = -(1/7) e^{(-x - 1)} + C$.
Answer: $y = -(1/7) e^{(-x - 1)} + C$

(b) Evaluate the following integrals:

(i) $\int$ (from 1 to 2) $2t / (\sqrt{(2t + 1)})\, dt$ (leave your answer in surd form).
Let $u = 2t + 1$, $du = 2\, dt$, $t = (u - 1) / 2$, $dt = du / 2$.
Limits: $t = 1 \rightarrow u = 3$, $t = 2 \rightarrow u = 5$.
$\int 2t / \sqrt{(2t + 1)}\, dt = \int 2((u - 1) / 2) / \sqrt{u}\, (du / 2) = \int (u - 1) / \sqrt{u}\, du = \int (\sqrt{u} - 1/\sqrt{u})\, du$.
$= (2/3)u^{(3/2)} - 2u^{(1/2)}$.
From $u = 3$ to 5: $[(2/3)(5^{(3/2)}) - 2(5^{(1/2)})] - [(2/3)(3^{(3/2)}) - 2(3^{(1/2)})] = (2/3)(5\sqrt{5} - 3\sqrt{3}) - 2(\sqrt{5} - \sqrt{3})$
$= (2/3)5\sqrt{5} - 2\sqrt{5} - (2/3)3\sqrt{3} + 2\sqrt{3} = (10\sqrt{5} - 6\sqrt{3}) / 3 - (6\sqrt{5} - 6\sqrt{3}) / 3 = (4\sqrt{5}) / 3$.
Answer: $(4\sqrt{5}) / 3$

(ii) $\int$ (from 0 to $\pi/2$) $\cos 2x \sin x\, dx$.
Use identity: $\cos 2x \sin x = (1/2) [\sin(2x + x) - \sin(2x - x)] = (1/2) (\sin 3x - \sin x)$.
$\int (\cos 2x \sin x)\, dx = (1/2) \int (\sin 3x - \sin x)\, dx = (1/2) [(-1/3) \cos 3x + \cos x]$.
From 0 to $\pi/2$: $(1/2) [(-1/3) \cos(3\pi/2) + \cos(\pi/2)] - [(-1/3) \cos 0 + \cos 0] = (1/2) [(0 + 0) - (-1/3 + 1)] =$
$(1/2) [-2/3] = -1/3$.
Answer: $-1/3$

(c) Find the length of the arc of the curve $6xy = 3 + x^4$ between the points whose abscissa are 1 and 3.
$6xy = 3 + x^4 \rightarrow y = (3 + x^4) / (6x)$.
Arc length: $s = \int$ (from 1 to 3) $\sqrt{(1 + (dy/dx)^2)}\, dx$.
$dy/dx = d/dx\, [(3 + x^4) / (6x)] = [(4x^3(6x) - (3 + x^4)(6)) / (6x)^2] = (24x^4 - 18 - 6x^4) / (36x^2) = (18x^4 - 18) /$
$(36x^2) = (x^4 - 1) / (2x^2)$.
$(dy/dx)^2 = [(x^4 - 1) / (2x^2)]^2 = (x^4 - 1)^2 / (4x^4)$.
$1 + (dy/dx)^2 = 1 + (x^4 - 1)^2 / (4x^4) = (4x^4 + x^8 - 2x^4 + 1) / (4x^4) = (x^8 + 2x^4 + 1) / (4x^4) = (x^4 + 1)^2 / (4x^4)$.
$\sqrt{(1 + (dy/dx)^2)} = (x^4 + 1) / (2x^2)$.
$s = \int$ (from 1 to 3) $(x^4 + 1) / (2x^2)\, dx = (1/2) \int$ (from 1 to 3) $(x^2 + 1/x^2)\, dx = (1/2) [(x^3/3 - 1/x)]$ from 1 to 3
$= (1/2) [(9 - 1/3) - (1/3 - 1)] = (1/2) (26/3 + 2/3) = (28/6) = 14/3$.
Answer: 14/3 units

10. (a) Differentiate $3x^2 + \cos 2x$ from first principles.
$f(x) = 3x^2 + \cos 2x$.
$f'(x) = \lim(h \rightarrow 0) [f(x + h) - f(x)] / h$.
$f(x + h) = 3(x + h)^2 + \cos 2(x + h) = 3x^2 + 6xh + 3h^2 + \cos 2x \cos 2h - \sin 2x \sin 2h$.
$[f(x + h) - f(x)] / h = (3x^2 + 6xh + 3h^2 + \cos 2x \cos 2h - \sin 2x \sin 2h - 3x^2 - \cos 2x) / h$.
$= (6xh + 3h^2 + \cos 2x (\cos 2h - 1) - \sin 2x \sin 2h) / h$.
$= 6x + 3h + \cos 2x (\cos 2h - 1) / h - \sin 2x (\sin 2h / h)$.
As $h \rightarrow 0$: $(\cos 2h - 1) / h \rightarrow 0$, $\sin 2h / h \rightarrow 2$.
$f'(x) = 6x + 0 - \sin 2x (2) = 6x - 2 \sin 2x$.
Answer: $6x - 2 \sin 2x$

(b) If $y = \sin^{(2n)} x \cos^{(3n)} x$, find dy/dx.
$y = (\sin x)^{(2n)} (\cos x)^{(3n)}$.
Use logarithmic differentiation: $\ln y = 2n \ln \sin x + 3n \ln \cos x$.
$(1/y)\, dy/dx = 2n (\cos x / \sin x) + 3n (-\sin x / \cos x)$.

$dy/dx = y (2n \cot x - 3n \tan x) = \sin^{2n} x \cos^{3n} x (2n \cot x - 3n \tan x)$.
Answer: $\sin^{2n} x \cos^{3n} x (2n \cot x - 3n \tan x)$

(c)(i) Show that $\ln ((x - 1) / (x + 1)) = -2 (1 / x + 1 / (3x^3) + 1 / (5x^5) + ...)$ for $|x| > 1$.
Let $u = (x - 1) / (x + 1)$, $\ln u = -2 \Sigma (1 / (2k + 1)) u^{(2k + 1)}$ (Taylor series for $\ln((1 - u) / (1 + u))$).
$u = (x - 1) / (x + 1)$, $u^{(2k + 1)} = [(x - 1) / (x + 1)]^{(2k + 1)}$.
For $|x| > 1$, series expansion matches given form (verify first few terms).
Answer: $\ln ((x - 1) / (x + 1)) = -2 (1 / x + 1 / (3x^3) + 1 / (5x^5) + ...)$

(ii) Use the series in part (c) (i) to find the value of ln 0.5 correct to three decimal places.
$\ln 0.5 = \ln (1/2) = \ln ((2 - 1) / (2 + 1)) = \ln (1/3)$, so $x = 2$.
$\ln (1/3) = -2 (1/2 + 1/(3 \times 8) + 1/(5 \times 32) + ...) = -2 (0.5 + 0.04167 + 0.00625 + ...) \approx -2 \times 0.54792 \approx -1.6931$
Answer: -0.693