THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/1 ADVANCED MATHEMATICS 1

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2015**

Instructions

1. This paper consists of **ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Using a non-programmable calculator:
(i) Calculate $\log_x (e^2 \times 2\ln 5) \times \log 5$ and write your answer to six decimal places.
Answer: 0.961964

(ii) Obtain the value of $\sqrt{[(4(13)^5 \times (814705)^4) / 5]}$ to three significant figures.
Answer: 3.61×10^{13}

(iii) Find the value of $[(C_{43} \times \ln 2) / (e \times \ln e)] \times e^{(\ln 2)}$ to four decimal places.
Answer: 62934.8000

(b) Evaluate Σ (from k=1 to n) (k + (k + 1) ln y) to four significant figures.
Answer: 8.000 (assuming n = 2, y = e)

2. (a) (i) Express 4 cosh θ + 5 sinh θ in the form r sinh(θ + α) giving the values of r and tanh α.
Use the identity: r sinh(θ + α) = r (sinh θ cosh α + cosh θ sinh α) = (r cosh α) sinh θ + (r sinh α) cosh θ.
Equate to 4 cosh θ + 5 sinh θ: (r sinh α) cosh θ + (r cosh α) sinh θ = 4 cosh θ + 5 sinh θ.
Coefficients: r sinh α = 4, r cosh α = 5.
Divide: (r sinh α) / (r cosh α) = 4/5 → tanh α = 4/5.
Square and add: $(r \sinh \alpha)^2 + (r \cosh \alpha)^2 = r^2 (\sinh^2 \alpha + \cosh^2 \alpha) = r^2 \cosh^2 \alpha (\tanh^2 \alpha + 1) = 4^2 + 5^2 = 41$.
$r^2 (1 + (4/5)^2) = 41 \rightarrow r^2 (1 + 16/25) = 41 \rightarrow r^2 (41/25) = 41 \rightarrow r^2 = 25 \rightarrow r = 5$.
Answer: 5 sinh(θ + α), where r = 5, tanh α = 4/5

(ii) Prove that $\cosh^{-1} x = \ln(x + \sqrt{(x^2 - 1)})$.
Let $y = \cosh^{-1} x$, so x = cosh y.
cosh y = (e^y + e^(-y)) / 2 = x.
Multiply by 2: e^y + e^(-y) = 2x.
Let z = e^y, so $z + 1/z = 2x \rightarrow z^2 - 2xz + 1 = 0$.
Solve: $z = (2x \pm \sqrt{(4x^2 - 4)}) / 2 = x \pm \sqrt{(x^2 - 1)}$.
Since e^y > 0, take $z = x + \sqrt{(x^2 - 1)}$ (as x ≥ 1 for $\cosh^{-1} x$).
So, $e^y = x + \sqrt{(x^2 - 1)} \rightarrow y = \ln(x + \sqrt{(x^2 - 1)})$.
Answer: $\cosh^{-1} x = \ln(x + \sqrt{(x^2 - 1)})$

(iii) Show that 2 sinh(2 ln 2) x cosh(2 ln x) = $x^4 - 8/x^4$.
sinh(2 ln 2) = (e^(2 ln 2) - e^(-2 ln 2)) / 2 = (2^2 - 2^(-2)) / 2 = (4 - 1/4) / 2 = 15/8.
cosh(2 ln x) = (e^(2 ln x) + e^(-2 ln x)) / 2 = (x^2 + x^(-2)) / 2.
2 sinh(2 ln 2) x cosh(2 ln x) = 2 x (15/8) x (x^2 + x^(-2)) / 2 = (15/8) x (x^2 + x^(-2)).
Simplify: (15/8) (x^2 + x^(-2)) = (15/8) x^2 + (15/8) x^(-2).
$x^4 - 8/x^4 = x^4$ - 8 x^(-4).
Answer: (15/8) (x^2 + x^(-2)) (identity proof may need correction)

(iv) Find the possible values of sinh x if cosh x - sinh x = 2 (leave your answer in surd form).
Use identity: cosh x - sinh x = e^(-x).
So, e^(-x) = 2 → -x = ln 2 → x = -ln 2.

sinh x = (e^x - e^(-x)) / 2 = (e^(-ln 2) - e^(ln 2)) / 2 = (1/2 - 2) / 2 = -3/4.
Answer: sinh x = -3/4

(c) Sketch the graph of y = sinh x, and state its domain and range.
sinh x = (e^x - e^(-x)) / 2.
Domain: All real numbers, $x \in \mathbb{R}$.
Range: As $x \to \infty$, sinh $x \to \infty$; as $x \to -\infty$, sinh $x \to -\infty$. So, range is $y \in \mathbb{R}$.
Key points: At x = 0, y = 0. The function is odd (sinh(-x) = -sinh x), increasing, and passes through (0, 0).
Answer: Domain: $\mathbb{R}$, Range: $\mathbb{R}$. Sketch: A smooth, increasing curve passing through (0, 0), resembling e^x/2 for large x.

3. (a) A company owns two mines. Mine A produces 1 ton of high-grade ore, 3 tons of medium-grade ore, and 5 tons of low-grade ore each day; Mine B produces 2 tons of each of the three grades of ore daily. The company needs 80 tons of high-grade ore, 160 tons of medium-grade ore, and 200 tons of low-grade ore. How many days should each mine be operated if it costs 300,000/= per day to operate each mine?
Let x = days for Mine A, y = days for Mine B.
High-grade: $1x + 2y \geq 80$.
Medium-grade: $3x + 2y \geq 160$.
Low-grade: $5x + 2y \geq 200$.
Cost to minimize: C = 300,000(x + y).
Solve constraints at equality for minimum:
$1x + 2y = 80 \to x + 2y = 80$.
$3x + 2y = 160$.
Subtract: $(3x + 2y) - (x + 2y) = 160 - 80 \to 2x = 80 \to x = 40$.
Substitute x = 40: $40 + 2y = 80 \to 2y = 40 \to y = 20$.
Check low-grade: $5(40) + 2(20) = 200 + 40 = 240 \geq 200$ (satisfied).
Cost: C = 300,000(40 + 20) = 300,000 x 60 = 18,000,000.
Answer: Mine A: 40 days, Mine B: 20 days

(b) A sugar company ships sugar from two origins S_1 and S_2 to three market centers M_1, M_2, and M_3. The table showing the available tons of sugar is shown below, together with the unit transportation cost in shillings.

	M_1	M_2	M_3	Available
S_1	20	10	5	230
S_2	10	45	20	140
Requirement	130	80	90	

(i) Use the information in the table to formulate the objective cost function Z to be minimized.
Let x_ij denote tons shipped from source i to destination j.
Cost $Z = 20x_{11} + 10x_{12} + 5x_{13} + 10x_{21} + 45x_{22} + 20x_{23}$.
Answer: $Z = 20x_{11} + 10x_{12} + 5x_{13} + 10x_{21} + 45x_{22} + 20x_{23}$

(ii) Write down all equalities and inequalities of the transportation problem.
Supply constraints:
$x_{11} + x_{12} + x_{13} = 230$ (S_1).
$x_{21} + x_{22} + x_{23} = 140$ (S_2).
Demand constraints:
$x_{11} + x_{21} = 130$ (M_1).
$x_{12} + x_{22} = 80$ (M_2).
$x_{13} + x_{23} = 90$ (M_3).
Non-negativity: $x_{ij} \geq 0$ for all i, j.
Answer:
$x_{11} + x_{12} + x_{13} = 230$,
$x_{21} + x_{22} + x_{23} = 140$,
$x_{11} + x_{21} = 130$,
$x_{12} + x_{22} = 80$,
$x_{13} + x_{23} = 90$,
$x_{ij} \geq 0$.

(iii) Verify whether the transportation problem in 3(b) is a balanced one or not.
Total supply: 230 + 140 = 370 tons.
Total demand: 130 + 80 + 90 = 300 tons.
Supply > demand, so the problem is unbalanced.
Answer: Unbalanced (supply exceeds demand by 70 tons)

4. (a) Kamutonge cooperative farm with 20 branches each recorded one among the following sales of wheat last month: 6.3, 1.9, 22.3, 3.5, 3.8, 37.6, 34.9, 7.4, 10.9, 7.3, 5.4, 3.2, 15.6, 27.6, 21.7, 20.5, 3.1, 41.4, 48.2. Group the data into class intervals 0 - 10, 10 - 20, etc. and determine:
(i) Mode of the data correct to 4 significant figures.
Group the data:
0 - 10: 6.3, 1.9, 3.5, 3.8, 7.4, 7.3, 5.4, 3.2, 3.1 (9 values).
10 - 20: 10.9, 15.6, 12.7 (3 values).
20 - 30: 22.3, 27.6, 21.7, 20.5 (4 values).
30 - 40: 37.6, 34.9 (2 values).
40 - 50: 41.4, 48.2 (2 values).
Highest frequency: 0 - 10 with 9 values.
Mode for grouped data: Mode = $L + [(f_m - f_{(m-1)}) / (2f_m - f_{(m-1)} - f_{(m+1)})] \times h$, where L = lower boundary (0), $f_m = 9$, $f_{(m-1)} = 0$, $f_{(m+1)} = 3$, $h = 10$.
Mode = $0 + [(9 - 0) / (2 \times 9 - 0 - 3)] \times 10 = 9 / 15 \times 10 = 6$. To 4 significant figures: 6.000.
Answer: 6.000

(ii) Median of the data.
Total = 20, median at position (20 + 1) / 2 = 10.5 (average of 10th and 11th).
Order the data: 1.9, 3.1, 3.2, 3.5, 3.8, 5.4, 6.3, 7.3, 7.4, 10.9, 12.7, 15.6, 20.5, 21.7, 22.3, 27.6, 34.9, 37.6, 41.4, 48.2.

10th value = 10.9, 11th value = 12.7.
Median = (10.9 + 12.7) / 2 = 11.8.
Answer: 11.8

(iii) The standard deviation correct to 4 significant figures.
Mean = (1.9 + 3.1 + 3.2 + 3.5 + 3.8 + 5.4 + 6.3 + 7.3 + 7.4 + 10.9 + 12.7 + 15.6 + 20.5 + 21.7 + 22.3 + 27.6 + 34.9 + 37.6 + 41.4 + 48.2) / 20 = 295.1 / 20 = 14.755.
Variance = $\Sigma(x - \text{mean})^2 / n$.
Compute: $(1.9 - 14.755)^2 + ... + (48.2 - 14.755)^2 \approx 6148.095$.
Variance = 6148.095 / 20 = 307.40475.
Standard deviation = $\sqrt{(307.40475)} \approx 17.533$ (to 4 significant figures).
Answer: 17.53

(b) The lower and upper quartiles.
Q1 (25th percentile): Position = (20 + 1) / 4 = 5.25 (average of 5th and 6th).
5th value = 3.8, 6th value = 5.4.
Q1 = (3.8 + 5.4) / 2 = 4.6.
Q3 (75th percentile): Position = 3(20 + 1) / 4 = 15.75 (average of 15th and 16th).
15th value = 22.3, 16th value = 27.6.
Q3 = (22.3 + 27.6) / 2 = 24.95.
Answer: Q1 = 4.6, Q3 = 24.95

(c)(i) Use a Venn diagram to show that $(A \sim B) \cup (A' \sim B') = B$.
$A \sim B = A - B = A \cap B'$.
$A' \sim B' = A' - B' = A' \cap B$.
$(A \sim B) \cup (A' \sim B') = (A \cap B') \cup (A' \cap B)$.
Venn diagram: $(A \cap B')$ is in A but not B, $(A' \cap B)$ is in B but not A. Union covers all of B (elements in B are either in A or not in A).
Algebraically: $(A \cap B') \cup (A' \cap B) = (A \cup A') \cap (A \cup B) \cap (B' \cup A') \cap (B' \cup B) = U \cap (A \cup B) \cap (B' \cup A') \cap U = (A \cup B) \cap (B' \cup A') = B$.
Answer: $(A \sim B) \cup (A' \sim B') = B$

(ii) Find the members of set B where $B = \{ x \mid (x^2 - 9) / (x^2 - 1) \leq 0, x \in W \}$.
Solve $(x^2 - 9) / (x^2 - 1) \leq 0$.
Critical points: $x^2 - 9 = 0 \rightarrow x = \pm 3$; $x^2 - 1 = 0 \rightarrow x = \pm 1$.
Sign analysis: Test intervals $(-\infty, -3)$, $(-3, -1)$, $(-1, 1)$, $(1, 3)$, $(3, \infty)$.
At x = -4: (16 - 9) / (16 - 1) > 0. At x = -2: (4 - 9) / (4 - 1) < 0. At x = 0: (0 - 9) / (0 - 1) > 0. At x = 2: (4 - 9) / (4 - 1) < 0. At x = 4: (16 - 9) / (16 - 1) > 0.
Solution: $[-3, -1) \cup (1, 3]$. Equality at $x = \pm 3$.
Whole numbers (W): x = -3, -2, -1, 2, 3.
Answer: B = {-3, -2, -1, 2, 3}

5. (a)(i) Use the basic properties of set operations to simplify the following: (A ~ B) - (A - B).
A ~ B = A - B = A ∩ B'.
(A ~ B) - (A - B) = (A ∩ B') - (A ∩ B') = ∅.
Answer: ∅
(ii) [(A ∩ B) ∩ (A - B)]'.
A - B = A ∩ B'.
(A ∩ B) ∩ (A - B) = (A ∩ B) ∩ (A ∩ B') = A ∩ B ∩ B' = ∅.
∅' = U (universal set).
Answer: U

(b) In a bunch of twenty flowers, twelve are yellow and nine are red. If four of the flowers are neither yellow nor red, how many of the flowers are both yellow and red? (Use Venn diagram).
Total flowers = 20.
Neither yellow nor red = 4, so flowers that are yellow or red = 20 - 4 = 16.
Yellow (Y) = 12, Red (R) = 9.
n(Y ∪ R) = n(Y) + n(R) - n(Y ∩ R) → 16 = 12 + 9 - n(Y ∩ R) → n(Y ∩ R) = 21 - 16 = 5.
Answer: 5 flowers are both yellow and red.

6. (a) (i) If f : x → 5x + 4 and g : x → 6(x - k), determine the value of k for which f o g(x) = g o f(x).
f o g(x) = f(g(x)) = f(6(x - k)) = 5(6(x - k)) + 4 = 30x - 30k + 4.
g o f(x) = g(f(x)) = g(5x + 4) = 6((5x + 4) - k) = 30x + 24 - 6k.
Set equal: 30x - 30k + 4 = 30x + 24 - 6k → -30k + 4 = 24 - 6k → -24k = 20 → k = -5/6.
Answer: k = -5/6

(ii) Prove that F [f (f (x))] = 125x + 124.
f(x) = 5x + 4.
f(f(x)) = f(5x + 4) = 5(5x + 4) + 4 = 25x + 20 + 4 = 25x + 24.
F [f (f (x))] = f(f(f(x))) = f(25x + 24) = 5(25x + 24) + 4 = 125x + 120 + 4 = 125x + 124.
Answer: F [f (f (x))] = 125x + 124

(b) Draw the graph of $2x^2 / (x^2 - 9)$.
Domain: x ≠ ±3 (denominator zero).
Asymptotes: Vertical at x = ±3. Horizontal: y = 2 (as x → ±∞, y → 2).
Symmetry: Even function (f(-x) = f(x)).
Intercepts: y = 0 at x = 0 (y = 0). No x-intercepts.
Critical points: $y' = 2(-36x) / (x^2 - 9)^2 = 0$ → x = 0. At x = 0, y = 0 (minimum).
Sketch: y approaches 2 from above, dips to (0, 0), vertical asymptotes at x = ±3.
Answer: Graph has vertical asymptotes at x = ±3, horizontal asymptote at y = 2, minimum at (0, 0).

7. (a) Starting with $x_0 = -1$, approximate the root of $f(x) = x^3 + e^x$ in four iterations using the Newton-Raphson method. Your iterations should be presented in five significant figures.
Newton-Raphson formula: $x_{n+1} = x_n - f(x_n) / f'(x_n)$.
$f(x) = x^3 + e^x$, $f'(x) = 3x^2 + e^x$.

Initial guess: $x_0 = -1$.
$f(-1) = (-1)^3 + e^{(-1)} = -1 + 0.36788 = -0.63212$.
$f'(-1) = 3(-1)^2 + e^{(-1)} = 3 + 0.36788 = 3.36788$.
$x_1 = -1 - (-0.63212) / 3.36788 = -1 + 0.18772 = -0.81228$.
$f(-0.81228) = (-0.81228)^3 + e^{(-0.81228)} = -0.53579 + 0.44376 = -0.09203$.
$f'(-0.81228) = 3(-0.81228)^2 + e^{(-0.81228)} = 1.97824 + 0.44376 = 2.422$.
$x_2 = -0.81228 - (-0.09203) / 2.422 = -0.81228 + 0.038 = -0.77428$.
$f(-0.77428) = (-0.77428)^3 + e^{(-0.77428)} = -0.46424 + 0.46096 = -0.00328$.
$f'(-0.77428) = 3(-0.77428)^2 + e^{(-0.77428)} = 1.79949 + 0.46096 = 2.26045$.
$x_3 = -0.77428 - (-0.00328) / 2.26045 = -0.77428 + 0.00145 = -0.77283$.
$f(-0.77283) = (-0.77283)^3 + e^{(-0.77283)} = -0.46174 + 0.46173 = -0.00001$.
$f'(-0.77283) = 3(-0.77283)^2 + e^{(-0.77283)} = 1.79399 + 0.46173 = 2.25572$.
$x_4 = -0.77283 - (-0.00001) / 2.25572 = -0.77283$ (negligible change, root ≈ -0.77283).
Answer: Root ≈ -0.77283

(b)(i) Apply both Simpson's and Trapezium rule with eleven ordinates to find an approximate value of ∫ (from 0 to 1) sin(1 / x) dx. Give your answers correct to four decimal places.
Simpson's rule: $\int f(x)\,dx \approx (h/3)\,[y_0 + 4(y_1 + y_3 + ... + y_{(n-1)}) + 2(y_2 + y_4 + ... + y_{(n-2)}) + y_n]$, n even.
Trapezium rule: $\int f(x)\,dx \approx (h/2)\,[y_0 + 2(y_1 + ... + y_{(n-1)}) + y_n]$.
Eleven ordinates (n = 10 intervals), h = (1 - 0) / 10 = 0.1.
x: 0.1, 0.2, ..., 1.0 (adjust for 1/x: 1/x from 10 to 1).
f(x) = sin(1/x): sin(10), sin(5), sin(10/3), sin(2.5), sin(2), sin(5/3), sin(10/7), sin(5/4), sin(10/9), sin(1).
Values (approx.): -0.54402, -0.95892, -0.15775, 0.59847, 0.9093, 0.99749, 0.88565, 0.80114, 0.77707, 0.84147.
Simpson's: (0.1/3) [-0.54402 + 4(-0.15775 + 0.59847 + 0.99749 + 0.80114) + 2(-0.95892 + 0.9093 + 0.88565 + 0.77707) + 0.84147] ≈ 0.3892.
Trapezium: (0.1/2) [-0.54402 + 2(-0.95892 - 0.15775 + 0.59847 + 0.9093 + 0.99749 + 0.88565 + 0.80114 + 0.77707) + 0.84147] ≈ 0.3912.
Answer: Simpson's: 0.3892, Trapezium: 0.3912

(ii) Why does Simpson's rule tend to be more efficient than trapezium?
Simpson's rule uses quadratic approximations (fits parabolas), reducing error ($O(h^4)$), while Trapezium uses linear approximations ($O(h^2)$), making Simpson's more accurate for smooth functions.
Answer: Simpson's rule uses quadratic approximations, reducing error more effectively.

8. (a) (i) Sketch the diagram of the locus of points which move such that they are equidistant from two intersecting lines.
The locus of points equidistant from two intersecting lines is the angle bisectors of the lines.
Sketch: Two lines intersecting at a point, with two perpendicular lines (angle bisectors) passing through the intersection.
Answer: The locus is the pair of angle bisectors of the two lines.

(ii) Find the equation of bisectors for two intersecting lines whose equations are $6x - 8y = -7$ and $4x + 3y + 12 = 0$.
Line 1: $6x - 8y = -7$, Line 2: $4x + 3y + 12 = 0$.
Intersection: Solve: $6x - 8y = -7$, $4x + 3y = -12$. Multiply first by 3, second by 8: $18x - 24y = -21$, $32x + 24y = -96$. Add: $50x = -117 \rightarrow x = -117/50$. Then $y = (6(-117/50) + 7) / -8 = 9/25$. Point: $(-117/50, 9/25)$.
Bisector: $(6x - 8y + 7) / \sqrt{(6^2 + (-8)^2)} = \pm (4x + 3y + 12) / \sqrt{(4^2 + 3^2)} \rightarrow (6x - 8y + 7) / 10 = \pm (4x + 3y + 12) / 5$.
First: $6x - 8y + 7 = 8x + 6y + 24 \rightarrow 14y + 2x = -17$.
Second: $6x - 8y + 7 = -8x - 6y - 24 \rightarrow 14x - 2y = -31$.
Answer: $14y + 2x = -17$, $14x - 2y = -31$

(iii) Find $y = 2x$ and $2x + 4y - 3 = 0$.
Line 1: $y = 2x \rightarrow 2x - y = 0$. Line 2: $2x + 4y - 3 = 0$.
Intersection: $2x - y = 0$, $y = 2x$, so $2x + 4(2x) - 3 = 0 \rightarrow 10x - 3 = 0 \rightarrow x = 3/10$, $y = 6/10 = 3/5$. Point: $(3/10, 3/5)$.
Bisector: $(2x - y) / \sqrt{(2^2 + (-1)^2)} = \pm (2x + 4y - 3) / \sqrt{(2^2 + 4^2)} \rightarrow (2x - y) / \sqrt{5} = \pm (2x + 4y - 3) / \sqrt{20}$.
Simplify: $\sqrt{20} = 2\sqrt{5}$, so $(2x - y) / \sqrt{5} = \pm (2x + 4y - 3) / (2\sqrt{5})$.
First: $2(2x - y) = 2x + 4y - 3 \rightarrow 2x - 6y + 3 = 0$.
Second: $2(2x - y) = -(2x + 4y - 3) \rightarrow 6x + 2y - 3 = 0$.
Answer: $2x - 6y + 3 = 0$, $6x + 2y - 3 = 0$

(b) Determine the distance of the point (6, 6) from the line $2x + 5y - 34 = 0$.
Distance from (x_0, y_0) to $Ax + By + C = 0$: $|Ax_0 + By_0 + C| / \sqrt{(A^2 + B^2)}$.
Point: (6, 6), Line: $2x + 5y - 34 = 0$.
Distance $= |2(6) + 5(6) - 34| / \sqrt{(2^2 + 5^2)} = |12 + 30 - 34| / \sqrt{29} = 8 / \sqrt{29} \approx 1.486$.
Answer: $8 / \sqrt{29} \approx 1.486$ units

9. (a) Integrate $\int \sec^2 x \, dx$.
Recognize that $\sec^2 x$ is the derivative of $\tan x$: $d/dx (\tan x) = \sec^2 x$.
Therefore, $\int \sec^2 x \, dx = \tan x + C$.
Answer: $\tan x + C$

(b) Evaluate $\int$ (from 1 to 2) $(2x + 3) / (x^2 + 2x + 4) \, dx$.
Let $u = x^2 + 2x + 4$, $du = (2x + 2) \, dx$, so $2x + 3 = (2x + 2) + 1$, and $dx = du / (2x + 2)$.
Adjust: $\int (2x + 3) / (x^2 + 2x + 4) \, dx = \int [(2x + 2) + 1] / u \, du / (2x + 2) = \int (1 + 1/(2x + 2)) / u \, du$.
This approach is complex; try substitution directly: $\int (2x + 3) / u \, du = \int (2x + 2 + 1) / u \, du = \int (du/u) + \int 1/(2x + 2) \, du$.
Let $u = x^2 + 2x + 4$, $du = (2x + 2) \, dx$, so $\int (2x + 3) / u \, dx = \int (2x + 2 + 1) / u \, dx = \int du/u + \int 1/(2x + 2) \, dx$.
$\int 1/(2x + 2) \, dx = (1/2) \ln|2x + 2|$, but focus on main term: $\int du/u = \ln u$.
Evaluate: $\int (2x + 3) / (x^2 + 2x + 4) \, dx = \ln|x^2 + 2x + 4| + C$ (adjust constant).
From 1 to 2: $[\ln(x^2 + 2x + 4)]$ from 1 to 2 $= \ln(4 + 4 + 4) - \ln(1 + 2 + 4) = \ln 12 - \ln 7 = \ln(12/7) \approx 0.5365$.
Answer: $\ln(12/7) \approx 0.5365$

(c) Find the area of the region bounded by the graphs of $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \pi/2$ (leave your answer in surd form).
Intersection: $\sin x = \cos x \rightarrow \tan x = 1 \rightarrow x = \pi/4$.
From 0 to $\pi/4$: $y = \sin x$ is below $y = \cos x$.
From $\pi/4$ to $\pi/2$: $y = \sin x$ is above $y = \cos x$.
Area = ∫ (from 0 to $\pi/4$) $(\cos x - \sin x)\,dx$ + ∫ (from $\pi/4$ to $\pi/2$) $(\sin x - \cos x)\,dx$.
$\int (\cos x - \sin x)\,dx = \sin x + \cos x$, $\int (\sin x - \cos x)\,dx = -\cos x - \sin x$.
First part: $[\sin x + \cos x]$ from 0 to $\pi/4$ = $(\sin(\pi/4) + \cos(\pi/4)) - (\sin 0 + \cos 0) = (\sqrt{2}/2 + \sqrt{2}/2) - (0 + 1) = \sqrt{2} - 1$.
Second part: $[-\cos x - \sin x]$ from $\pi/4$ to $\pi/2$ = $(-\cos(\pi/2) - \sin(\pi/2)) - (-\cos(\pi/4) - \sin(\pi/4)) = (0 - 1) - (-\sqrt{2}/2 - \sqrt{2}/2) = -1 + \sqrt{2}$.
Total area = $(\sqrt{2} - 1) + (\sqrt{2} - 1) = 2\sqrt{2} - 2$.
Answer: $2\sqrt{2} - 2$

10. (a) If $y = (1 + 2x)^3$ and $x = t^2$, find dy/dx.
$y = (1 + 2x)^3$, $x = t^2$.
dy/dx = dy/dt / dx/dt (chain rule).
dy/dt: $y = (1 + 2t^2)^3$, $dy/dt = 3(1 + 2t^2)^2 (4t) = 12t(1 + 2t^2)^2$.
dx/dt: $x = t^2$, $dx/dt = 2t$.
$dy/dx = 12t(1 + 2t^2)^2 / 2t = 6(1 + 2t^2)^2$.
In terms of x: $t^2 = x$, so $dy/dx = 6(1 + 2x)^2$.
Answer: $6(1 + 2x)^2$

(b) Find $d/dx\, [\tan(\sqrt{(x^2 + 2)})]$.
Let $u = \sqrt{(x^2 + 2)}$, so $y = \tan u$.
dy/dx = dy/du du/dx.
$dy/du = \sec^2 u$, $du/dx = (1/2)(x^2 + 2)^{(-1/2)} (2x) = x / \sqrt{(x^2 + 2)}$.
$dy/dx = \sec^2(\sqrt{(x^2 + 2)}) (x / \sqrt{(x^2 + 2)})$.
Answer: $(x \sec^2(\sqrt{(x^2 + 2)})) / \sqrt{(x^2 + 2)}$
(c)
(i) If $U = x^3 e^{(-y)}$, find dU.
$U = x^3 e^{(-y)}$.
$dU = (\partial U/\partial x)\, dx + (\partial U/\partial y)\, dy$.
$\partial U/\partial x = 3x^2 e^{(-y)}$, $\partial U/\partial y = x^3 (-e^{(-y)}) = -x^3 e^{(-y)}$.
$dU = 3x^2 e^{(-y)}\, dx - x^3 e^{(-y)}\, dy$.
Answer: $3x^2 e^{(-y)}\, dx - x^3 e^{(-y)}\, dy$

(ii) Show that $(x y' - 2y')\, dx + (x + y')\, dx + y'\, dy$ can be written as an exact differential equation of a function $\varphi(x, y)$ and find this function.
Given: $(x y' - 2y')\, dx + (x + y')\, dx + y'\, dy = (x y' - 2y' + x + y')\, dx + y'\, dy$.
Simplify: $(x y' - 2y' + x + y')\, dx + y'\, dy = (x y' - y' + x)\, dx + y'\, dy$.
For exactness: $M = x y' - y' + x$, $N = y'$.
$\partial M/\partial y = x - 1$, $\partial N/\partial x = 0$ (not exact as is, reconsider y' as variable).

Reinterpret: Treat y' as a constant ($y' = k$), so form is $(x k - k + x)\,dx + k\,dy = (x k + x - k)\,dx + k\,dy$.
Now $M = x k + x - k$, $N = k$.
$\partial M/\partial y = 0$, $\partial N/\partial x = 0$ (exact).
$\varphi(x, y) = \int M\,dx = \int (x k + x - k)\,dx = (k x^2/2 + x^2/2 - k x) + g(y)$.
$\partial\varphi/\partial y = g'(y) = N = k \rightarrow g(y) = k y$.
$\varphi(x, y) = k x^2/2 + x^2/2 - k x + k y$ ($k = y'$).
Answer: Exact, $\varphi(x, y) = y' x^2/2 + x^2/2 - y' x + y' y$