THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/1 ADVANCED MATHEMATICS 1

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2016**

Instructions

1. This paper consists of **ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Using a scientific calculator, find the following correct to four decimal places:

(i) $\sqrt{(3.12 \times \log 5)^5 / (\cos(\pi/9) + \sin 46°)}$

Answer: 5.7299

(ii) $([e^{(\log 6)} \times 6 \times \sinh^{-1}(0.6972)] / [(\ln 3.5) \times (\cos 64.5°) \times (\tan 46°)]) \times (0.6467)^4$

Answer: 1.8958

1. (b) A rat has a mass 30 grams at birth. It reaches maturity in 3 months. The rate of growth is modeled by the equation $dm/dt = 120 \times (2.1985t - 3)^{-2}$, where m in grams is the mass of the rat, t months after birth. Use the scientific calculator to find the mass of the rat when fully grown.

ans: 1353.9 g

2. (a) If $t = \tanh^2(x/2)$, express sinh x and cosh x in terms of t.

$\tanh(x/2) = \sqrt{t}$

Let $u = x/2$, so $x = 2u$, $\tanh u = \sqrt{t}$

$\tanh u = \sinh u / \cosh u = \sqrt{t}$

$\sinh u = \sqrt{t} \times \cosh u$

$(\sinh u)^2 = (\cosh u)^2 - 1$

$(\sqrt{t} \times \cosh u)^2 = (\cosh u)^2 - 1$

$t \times (\cosh u)^2 = (\cosh u)^2 - 1$

$(t - 1) \times (\cosh u)^2 = -1$

$(\cosh u)^2 = 1 / (1 - t)$

$\cosh u = \sqrt{1 / (1 - t)}$

$\sinh u = \sqrt{t} \times \sqrt{1 / (1 - t)} = \sqrt{t / (1 - t)}$

$\sinh x = \sinh(2u) = 2 \times \sinh u \times \cosh u = 2 \times \sqrt{t / (1 - t)} \times \sqrt{1 / (1 - t)} = 2 \times \sqrt{t / (1 - t)^2} = 2 \times \sqrt{t} / (1 - t)$

cosh x = cosh(2u) = 2 x $(\cosh u)^2$ - 1 = 2 x (1 / (1 - t)) - 1 = (2 - (1 - t)) / (1 - t) = (1 + t) / (1 - t)

Answer: sinh x = 2 x √t / (1 - t), cosh x = (1 + t) / (1 - t)

(b) Express $\sinh^4 x$ - ln x in terms of natural logarithms; hence, find the limit as x → ∞.

sinh x = (e^x - e^(-x)) / 2

$(\sinh x)^4$ = $[(e^x - e^{-x}) / 2]^4$ = (1/16) x $(e^x - e^{-x})^4$

$(e^x - e^{-x})^4$ = $(e^x - e^{-x})^2$ x $(e^x - e^{-x})^2$ = $(e^{2x} - 2 + e^{-2x})^2$ = e^(4x) + e^(-4x) - 4 x e^(2x) + 4 - 4 x e^(-2x) + e^(-4x)

So, $(\sinh x)^4$ = (1/16) x (e^(4x) + e^(-4x) - 4 x e^(2x) + 4 - 4 x e^(-2x) + e^(-4x))

$\sinh^4 x$ - ln x = (1/16) x (e^(4x) + 2 x e^(-4x) - 4 x e^(2x) - 4 x e^(-2x) + 4) - ln x

As x → ∞, e^(-4x), e^(-2x) → 0, so $(\sinh x)^4$ ≈ (1/16) x e^(4x), and ln x grows slower than e^(4x)

Limit: (1/16) x e^(4x) - ln x → ∞

Answer: (1/16) x (e^(4x) + 2 x e^(-4x) - 4 x e^(2x) - 4 x e^(-2x) + 4) - ln x; limit as x → ∞ is ∞

(c) Evaluate ∫(1 / [√($4x^2$ - 8x + 7)]) dx correct to four decimal places (assume limits 0 to 1 for definite integral).

Complete the square: $4x^2 - 8x + 7 = 4(x^2 - 2x) + 7 = 4((x - 1)^2 - 1) + 7 = 4(x - 1)^2 + 3$

∫(1 / √($4(x - 1)^2$ + 3)) dx

Let u = x - 1, du = dx

∫(1 / √($4u^2$ + 3)) du, from u = -1 to 0

∫(1 / √($4u^2$ + 3)) du = (1/2) x ∫(1 / √(u^2 + (3/4))) du

This is arcsinh form: ∫(1 / √($u^2 + a^2$)) du = arcsinh(u/a), where a = √(3/4) = √3/2

So, (1/2) x arcsinh(u / (√3/2)) = (1/2) x arcsinh(2u / √3)

Evaluate from u = -1 to 0: (1/2) x [arcsinh(0) - arcsinh(-2/√3)]

arcsinh(0) = 0, arcsinh(-2/√3) = -arcsinh(2/√3)

arcsinh(2/√3) = ln((2/√3) + √($(2/\sqrt{3})^2$ + 1)) = ln((2/√3) + √(7/3)) ≈ 0.8085

So, (1/2) x (0 - (-0.8085)) = 0.4043

Answer: 0.4043

3. (a) Mr. Mutu takes two types of vitamin pills. He must have at least 16 units of vitamin A, 5 units of vitamin B, and 20 units of vitamin C. He can choose between pill M which contains 5 units of A, 1 unit of B and 2 units of C, and pill N which contains 2 units of A, 1 unit of B and 7 units of C. Pill M costs 150 shillings and pill N costs 300 shillings. How many pills of each type should he buy in order to minimize the cost?

Let x = number of M pills, y = number of N pills

Constraints:

$5x + 2y \geq 16$ (vitamin A)

$x + y \geq 5$ (vitamin B)

$2x + 7y \geq 20$ (vitamin C)

$x \geq 0, y \geq 0$

Cost to minimize: $C = 150x + 300y$

Solve inequalities:

From $x + y \geq 5$, $y \geq 5 - x$

$5x + 2y \geq 16$: $5x + 2(5 - x) \geq 16 \rightarrow 3x + 10 \geq 16 \rightarrow x \geq 2$

$2x + 7y \geq 20$: $2x + 7(5 - x) \geq 20 \rightarrow -5x + 35 \geq 20 \rightarrow -5x \geq -15 \rightarrow x \leq 3$

So, $2 \leq x \leq 3$, and $y = 5 - x$ (from vitamin B constraint at equality for minimum)

Test x = 2: y = 5 - 2 = 3

Vitamin A: 5 x 2 + 2 x 3 = 10 + 6 = 16 (satisfied)

Vitamin C: 2 x 2 + 7 x 3 = 4 + 21 = 25 (satisfied)

Test x = 3: y = 5 - 3 = 2

Vitamin A: 5 x 3 + 2 x 2 = 15 + 4 = 19 (satisfied)

Vitamin C: 2 x 3 + 7 x 2 = 6 + 14 = 20 (satisfied)

Cost:

$x = 2, y = 3$: $C = 150 \times 2 + 300 \times 3 = 300 + 900 = 1200$

$x = 3, y = 2$: $C = 150 \times 3 + 300 \times 2 = 450 + 600 = 1050$

Answer: 3 M pills, 2 N pills

(b) A TV dealer has stores in Dar es Salaam and Morogoro and retailers in Tanga and Dodoma. The stores have a stock of 45 TV and 40 TV sets respectively while the retailers have requirements of the retailers are 25 and 30 TV sets respectively. If the cost of transporting a TV set from Dar es Salaam to Tanga is Tsh 5,000/= and from Dar es Salaam to Dodoma is Tsh 3,000/= and from Morogoro to Dodoma is Tsh 9,000/=, from Morogoro to Tanga is Tsh 6,000/=. How should the TV dealer supply the requested TV sets at minimum cost?

Let:

x = TVs from Dar to Tanga

y = TVs from Dar to Dodoma

$45 - x - y$ = TVs from Dar to Morogoro (but not needed)

z = TVs from Morogoro to Tanga

$40 - z$ = TVs from Morogoro to Dodoma

Constraints:

Tanga: $x + z = 25$

Dodoma: $y + (40 - z) = 30 \rightarrow y - z = -10$

$0 \leq x + y \leq 45, 0 \leq z \leq 40$

Solve:

$y - z = -10 \rightarrow y = z - 10$

$x + z = 25 \rightarrow x = 25 - z$

$x + y \leq 45$: $25 - z + z - 10 \leq 45 \rightarrow 15 \leq 45$ (always true)

$0 \leq z \leq 40, 0 \leq x, 0 \leq y$: $z - 10 \geq 0 \rightarrow z \geq 10$; $25 - z \geq 0 \rightarrow z \leq 25$

Cost: $C = 5000x + 3000y + 6000z + 9000(40 - z) = 5000(25 - z) + 3000(z - 10) + 6000z + 9000(40 - z)$

Simplify: $C = 125000 - 5000z + 3000z - 30000 + 6000z - 9000z + 360000 = 455000 - 5000z$

Minimize C: $z = 25$ (max within $10 \leq z \leq 25$)

$C = 455000 - 5000 \times 25 = 330000$

So: $x = 25 - 25 = 0$, $y = 25 - 10 = 15$, $z = 25$, $40 - z = 15$

Supply: Dar to Tanga: 0, Dar to Dodoma: 15, Morogoro to Tanga: 25, Morogoro to Dodoma: 15

Answer: Dar to Tanga: 0, Dar to Dodoma: 15, Morogoro to Tanga: 25, Morogoro to Dodoma: 15

(ii) What is the minimum cost?

From above: $C = 330000$

Answer: Tsh 330,000

4. (a) The frequency distribution of a variable X is classified into equal intervals of size C. The frequency in a class is denoted by f and the total frequencies is N. If the data is coded into a variable u by means of the relation x_bar = u x C + X_bar, where X_bar takes the central values of the class intervals, show that the standard deviation δ of the distribution is given by $\delta^2 = C^2 \times ((\Sigma f u^2 / N) - ((\Sigma f u / N))^2)$.

Step 1: Given the coding x_bar = u x C + X_bar, where x_bar is the original variable, X_bar is the class midpoint, and u is the coded variable.

Step 2: Solve for u: u = (x_bar - X_bar) / C.

Step 3: The standard deviation of x_bar is $\delta = \sqrt{(\Sigma f (x\_bar - x\_mean)^2) / N }$, where x_mean is the mean of x_bar.

Step 4: Compute the mean: x_mean = (Σ f x_bar) / N. Substitute x_bar = u x C + X_bar, so Σ f x_bar = Σ f (u x C + X_bar) = C x Σ f u + X_bar x Σ f. Since X_bar is the midpoint of each class, we need the overall mean.

Step 5: x_mean = (Σ f (u x C + X_bar)) / N = C x (Σ f u / N) + X_bar. If X_bar is adjusted per class, we simplify by centering: let X_bar be the reference point, often set such that Σ f u = 0 for simplicity in coding.

Step 6: Variance $\delta^2 = (\Sigma f (x\_bar - x\_mean)^2) / N$. Substitute x_bar - x_mean = (u x C + X_bar) - (C x (Σ f u / N) + X_bar) = C x (u - (Σ f u / N)).

Step 7: So, $(x\_bar - x\_mean)^2 = (C \times (u - (\Sigma f u / N)))^2 = C^2 \times (u - (\Sigma f u / N))^2$.

Step 8: $\delta^2 = (\Sigma f (C^2 \times (u - (\Sigma f u / N))^2)) / N = C^2 \times (\Sigma f (u - (\Sigma f u / N))^2) / N$.

Step 9: Expand: $\Sigma f (u - (\Sigma f u / N))^2 = \Sigma f u^2 - 2 \times (\Sigma f u / N) \times \Sigma f u + \Sigma f (\Sigma f u / N)^2 = \Sigma f u^2 - 2 \times (\Sigma f u)^2 / N + N \times (\Sigma f u / N)^2 = \Sigma f u^2 - (\Sigma f u)^2 / N$.

Step 10: Thus, $\delta^2 = C^2 \times ((\Sigma f u^2 / N) - ((\Sigma f u)^2 / N^2)) = C^2 \times ((\Sigma f u^2 / N) - ((\Sigma f u / N))^2)$.

Answer: $\delta^2 = C^2 \times ((\Sigma f u^2 / N) - ((\Sigma f u / N))^2)$, as required.

(b) The average heights of 20 boys and 30 girls are 160 cm and 155 cm respectively. If the corresponding standard deviation of boys and girls are 4 cm and 3.5 cm, find the standard deviation of the whole group.

Step 1: Boys: n1 = 20, mean1 = 160 cm, sd1 = 4 cm. Girls: n2 = 30, mean2 = 155 cm, sd2 = 3.5 cm.

Step 2: Total group: N = 20 + 30 = 50.

Step 3: Mean of the whole group: mean_total = (n1 x mean1 + n2 x mean2) / N = (20 x 160 + 30 x 155) / 50 = (3200 + 4650) / 50 = 7850 / 50 = 157 cm.

Step 4: Variance formula for combined groups: var_total = (n1 x (var1 + $(\text{mean1} - \text{mean\_total})^2$) + n2 x (var2 + $(\text{mean2} - \text{mean\_total})^2$)) / N.

Step 5: var1 = $(\text{sd1})^2 = 4^2 = 16$, var2 = $(\text{sd2})^2 = (3.5)^2 = 12.25$.

Step 6: $(\text{mean1} - \text{mean\_total})^2 = (160 - 157)^2 = 3^2 = 9$, $(\text{mean2} - \text{mean\_total})^2 = (155 - 157)^2 = (-2)^2 = 4$.

Step 7: var_total = (20 x (16 + 9) + 30 x (12.25 + 4)) / 50 = (20 x 25 + 30 x 16.25) / 50 = (500 + 487.5) / 50 = 987.5 / 50 = 19.75.

Step 8: Standard deviation = $\sqrt{(\text{var\_total})} = \sqrt{(19.75)} \approx 4.444$.

Answer: 4.44 cm (to two decimal places)

(c) The following table shows the length of 100 earth worms in millimetres:

Length (mm)	95 – 109	110 – 124	125 – 139	140 – 154	155 – 169	170 – 184	185 – 199	200 - 214
Number of worms	2	8	17	26	14	16	6	1

Obtain the semi-interquartile range correct to two significant figures.

Step 1: Total worms N = 2 + 8 + 17 + 26 + 14 + 16 + 6 + 1 = 100.

Step 2: Q1 (25th percentile): 25th worm. Cumulative frequencies: 2, 10, 27, ... Q1 lies in 125 - 139 class (17 worms, cumulative 10 to 27).

Step 3: Q1 position: (25 - 10) / 17 = 15/17 into the class. Class width = 15 mm, lower boundary = 124.5 mm.

Step 4: Q1 = 124.5 + (15/17) x 15 ≈ 124.5 + 13.24 = 137.74 mm.

Step 5: Q3 (75th percentile): 75th worm. Cumulative frequencies: ..., 53, 67, 83, 89, ... Q3 lies in 170 - 184 class (16 worms, cumulative 67 to 83).

Step 6: Q3 position: (75 - 67) / 16 = 8/16 = 0.5 into the class. Lower boundary = 169.5 mm.

Step 7: Q3 = 169.5 + 0.5 x 15 = 169.5 + 7.5 = 177 mm.

Step 8: Semi-interquartile range = (Q3 - Q1) / 2 = (177 - 137.74) / 2 = 39.26 / 2 = 19.63.

Step 9: To two significant figures: 20.

Answer: 20 mm

5. (a) Use the laws of algebra of sets to:

(i) Verify that $X \cup (X \cap Y) = X$

Step 1: $X \cap Y$ is the intersection, so $X \cup (X \cap Y)$ means X union with the elements common to X and Y.

Step 2: Since $X \cap Y \subseteq X$ (intersection is a subset of X), adding $X \cap Y$ to X via union doesn't add anything beyond X.

Step 3: Formally: Let $z \in X \cup (X \cap Y)$. Then $z \in X$ or $z \in (X \cap Y)$. If $z \in X$, already in X. If $z \in (X \cap Y)$, then $z \in X$ and $z \in Y$, so $z \in X$. Thus, $X \cup (X \cap Y) \subseteq X$.

Step 4: Conversely, if $z \in X$, then $z \in X \cup (X \cap Y)$. So $X \subseteq X \cup (X \cap Y)$.

Step 5: Hence, $X \cup (X \cap Y) = X$.

Answer: Verified.

(ii) Simplify $[A \cap (A \cup B)']'$

Step 1: Simplify inside the brackets. $(A \cup B)'$ is the complement of $(A \cup B)$, which by De Morgan's laws is $A' \cap B'$.
Step 2: So, $A \cap (A \cup B)' = A \cap (A' \cap B') = (A \cap A') \cap B' = \emptyset \cap B' = \emptyset$ (since $A \cap A' = \emptyset$).
Step 3: Now take the complement: $[A \cap (A \cup B)']' = \emptyset' = U$ (the universal set).

(b) If A, B and C are three non-empty sets, use Venn diagram to show whether $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

Step 1: The formula is the inclusion-exclusion principle for three sets.

Step 2: Consider a Venn diagram with three overlapping circles A, B, C.

Step 3: $n(A \cup B \cup C)$ counts all elements in at least one set.

Step 4: $n(A) + n(B) + n(C)$ counts all elements but overcounts intersections: $A \cap B$, $A \cap C$, $B \cap C$ are counted twice, $A \cap B \cap C$ is counted thrice.

Step 5: Subtract $n(A \cap B)$, $n(A \cap C)$, $n(B \cap C)$ to correct double-counting.

Step 6: But $A \cap B \cap C$ was subtracted thrice (once for each pair), and was originally counted thrice, so add back $n(A \cap B \cap C)$.

Step 7: Formula holds: $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

(c) A class contains 15 boys and 15 girls. A survey of the class showed that: 20 pupils were studying Geography, 14 pupils were studying Geography, 10 of the girls were studying Mathematics, 4 of the girls were studying both Geography and Mathematics, 3 of the boys were studying neither Geography nor Mathematics. How many pupils were studying both Mathematics and Geography?

Step 1: Total pupils = 15 boys + 15 girls = 30.

Step 2: "14 pupils were studying Geography" seems redundant or incorrect since 20 were already stated. Assume 14 pupils were studying Mathematics (contextual correction).

Step 3: Given: Geography (G) = 20 pupils, Mathematics (M) = 14 pupils, 10 girls in M, 4 girls in $G \cap M$, 3 boys in neither G nor M.

Step 4: Boys in G or M = 15 - 3 = 12.

Step 5: Girls: 10 in M, 4 in $G \cap M$. Girls in M only = 10 - 4 = 6. Let girls in G only = x. Total girls = 15.

Step 6: Total G = 20, $G \cap M$ for girls = 4, let $G \cap M$ for boys = y. Total $G \cap M$ = 4 + y.

Step 7: Total M = 14, M for girls = 10, so M for boys = 14 - 10 = 4.

Step 8: Use Venn diagram for boys: Boys in M = 4, boys in $G \cap M$ = y, boys in G only = 12 - 4 = 8 (adjust later).

Step 9: Total G = 20: Girls in G only + 4 + boys in G only + y = 20.

Step 10: Total pupils = 30: (G only) + (M only) + ($G \cap M$) + (neither) = 30.

Step 11: Solve: Total G ∩ M = 4 (girls) + y (boys). From M: 6 (girls M only) + 4 (boys M only) + 4 + y = 14, which holds.

Step 12: Neither = 3 (all boys). Girls: x (G only) + 6 (M only) + 4 (G ∩ M) + neither = 15, so x + 10 = 15, x = 5.

Step 13: Boys: G only + M only + G ∩ M + neither = 15. M only = 4 - y, G only = (20 - 5 - 4 - y) = 11 - y, so (11 - y) + (4 - y) + y + 3 = 15, 18 - y = 15, y = 3.

Step 14: Total G ∩ M = 4 + 3 = 7.

Answer: 7 pupils were studying both Mathematics and Geography.

6. (a) Use the table of values to draw the graph of f(x) = 2 x e^x + x - 3 x 5 ≤ 1.2 and g(x) = 1 - e^x + x - 3.5 x ≤ 2.7 on the same xy plane.

The problem asks to draw the graphs, but no table of values is provided. Since I can't generate or draw graphs directly, I'll describe how to plot them.

For f(x) = 2 x e^x + x - 3, x from 5 to 1.2 seems incorrect (likely a typo, as 5 to 1.2 decreases). Assume x from -3.5 to 1.2 (based on context of g(x)).

For g(x) = 1 - e^x + x - 3.5, x from -3.5 to 2.7.

Adjust the range: let's use x from -3.5 to 1.2 for both (common range for comparison).

Compute key points for f(x):

x = -3.5: e^(-3.5) ≈ 0.0302, f(-3.5) = 2 x 0.0302 + (-3.5) - 3 ≈ -6.44

x = -2: e^(-2) ≈ 0.1353, f(-2) = 2 x 0.1353 + (-2) - 3 ≈ -4.73

x = 0: e^0 = 1, f(0) = 2 x 1 + 0 - 3 = -1

x = 1: e^1 ≈ 2.718, f(1) = 2 x 2.718 + 1 - 3 ≈ 3.44

x = 1.2: e^1.2 ≈ 3.32, f(1.2) = 2 x 3.32 + 1.2 - 3 ≈ 4.84

For g(x):

x = -3.5: g(-3.5) = 1 - 0.0302 + (-3.5) - 3.5 ≈ -6.03

x = -2: g(-2) = 1 - 0.1353 + (-2) - 3.5 ≈ -4.64

x = 0: g(0) = 1 - 1 + 0 - 3.5 = -3.5

x = 1: g(1) = 1 - 2.718 + 1 - 3.5 ≈ -4.22

x = 1.2: g(1.2) = 1 - 3.32 + 1.2 - 3.5 ≈ -4.62

6. (b) Given that, $f(x) = x + 1 - 1/x$ and that $g(x) = 1 - 1/x$:

(i) Write down the composite function g o f(x) in its simplest form.

$g \circ f(x) = g(f(x))$.

$f(x) = x + 1 - 1/x$.

$g(x) = 1 - 1/x$, so $g(f(x)) = 1 - 1/(f(x))$.

Substitute: $g(f(x)) = 1 - 1/(x + 1 - 1/x)$.

Simplify the denominator: $x + 1 - 1/x = (x^2 + x - 1)/x$.

So, $1/(x + 1 - 1/x) = x/(x^2 + x - 1)$.

Thus, $g(f(x)) = 1 - x/(x^2 + x - 1) = (x^2 + x - 1 - x)/(x^2 + x - 1) = (x^2 - 1)/(x^2 + x - 1)$.

Answer: $g \circ f(x) = (x^2 - 1)/(x^2 + x - 1)$

(ii) Find the value of x if $g \circ f(x) = f \circ g(x)$.

$f \circ g(x) = f(g(x))$.

$g(x) = 1 - 1/x$, so $f(g(x)) = (1 - 1/x) + 1 - 1/(1 - 1/x)$.

Simplify: $1 - 1/x + 1 = 2 - 1/x$, and $1/(1 - 1/x) = x/(x - 1)$.

So, $f(g(x)) = 2 - 1/x - x/(x - 1) = (2(x - 1) - 1 - x)/(x - 1) = (2x - 2 - 1 - x)/(x - 1) = (x - 3)/(x - 1)$.

Set $g \circ f(x) = f \circ g(x)$: $(x^2 - 1)/(x^2 + x - 1) = (x - 3)/(x - 1)$.

Cross-multiply: $(x^2 - 1)(x - 1) = (x - 3)(x^2 + x - 1)$.

Expand: $(x^3 - x^2 - x + 1) = (x^3 + x^2 - x - 3x^2 - 3x + 3) = (x^3 - 2x^2 - 4x + 3)$.

Equate: $x^3 - x^2 - x + 1 = x^3 - 2x^2 - 4x + 3$.

Simplify: $x^2 + 3x - 2 = 0$.

Solve: $x = (-3 \pm \sqrt{(9 + 8)})/2 = (-3 \pm \sqrt{17})/2$.

Answer: $x = (-3 \pm \sqrt{17})/2$

(c) Find the equation of the asymptotes of the curve $y = (x^2 + 3)/(x - 1)$ and sketch the curve showing the coordinates of the turning points.

(i) Equation of the asymptotes:

Vertical asymptote: Denominator = 0, so $x - 1 = 0$, $x = 1$.

Horizontal/oblique asymptote: Divide numerator by denominator: $(x^2 + 3)/(x - 1) = x + 1 + 4/(x - 1)$. As $x \to \pm\infty$, $y \to x + 1$.

So, oblique asymptote: $y = x + 1$.

Answer: Vertical: $x = 1$, Oblique: $y = x + 1$

(ii) Turning points:

$y = (x^2 + 3)/(x - 1)$.

Use quotient rule: $dy/dx = [(2x)(x - 1) - (x^2 + 3)(1)]/(x - 1)^2 = (2x^2 - 2x - x^2 - 3)/(x - 1)^2 = (x^2 - 2x - 3)/(x - 1)^2$.

Set $dy/dx = 0$: $x^2 - 2x - 3 = 0 \to (x - 3)(x + 1) = 0 \to x = 3, x = -1$.

At $x = 3$: $y = (9 + 3)/(3 - 1) = 6$. Point: (3, 6).

At $x = -1$: $y = (1 + 3)/(-1 - 1) = -2$. Point: (-1, -2).

Sketch: Plot asymptotes, turning points, and note behavior (e.g., as $x \to 1^+$, $y \to \infty$; $x \to 1^-$, $y \to -\infty$).

Answer: Turning points: (3, 6) and (-1, -2)

7. (a) (i) Write down four sources of errors in numerical computations.

Truncation error: Approximating infinite processes (e.g., cutting off a series).

Round-off error: Finite precision in representing numbers.

Overflow/underflow: Numbers too large/small for the system.

Algorithmic error: Errors from the method’s approximations.

Answer: Truncation, round-off, overflow/underflow, algorithmic errors

(ii) If x_{num} is a better approximation to a root of the equation $f(x_{num}) = 0$. Derive the Newton-Raphson method for the function $f(x_{num})$.

Start with an initial guess x_0.

Taylor expansion: $f(x) \approx f(x_n) + f'(x_n)(x - x_n)$.

Set $f(x) = 0$: $0 = f(x_n) + f'(x_n)(x - x_n)$.

Solve for x: $x = x_n - f(x_n)/f'(x_n)$.

So, next approximation: $x_{(n+1)} = x_n - f(x_n)/f'(x_n)$.

Answer: $x_{(n+1)} = x_n - f(x_n)/f'(x_n)$

(b) Use the Newton-Raphson method obtained in (d) (iii) to derive the secant formula and comment why would you want to use it instead of the Newton-Raphson method.

Newton-Raphson: $x_{(n+1)} = x_n - f(x_n)/f'(x_n)$.

Secant method approximates $f'(x_n)$ using two points: $f'(x_n) \approx (f(x_n) - f(x_{(n-1)}))/(x_n - x_{(n-1)})$.

Substitute: $x_{(n+1)} = x_n - f(x_n) \times (x_n - x_{(n-1)})/(f(x_n) - f(x_{(n-1)}))$.

Why use secant? It doesn’t require computing the derivative, which can be complex or unavailable.

Answer: Secant: $x_{(n+1)} = x_n - f(x_n) \times (x_n - x_{(n-1)})/(f(x_n) - f(x_{(n-1)}))$; use it to avoid derivative computation.

(c) Using the secant method obtained in (e) with $x_0 = 2$ and $x_1 = 3$ perform three iterations to approximate the root of $x^2 - 2x - 1 = 0$ and hence compute the absolute error correct to four decimal places.

$f(x) = x^2 - 2x - 1$.

$x_0 = 2$, $x_1 = 3$.

$f(2) = 4 - 4 - 1 = -1$, $f(3) = 9 - 6 - 1 = 2$.

Iteration 1: $x_2 = x_1 - f(x_1) \times (x_1 - x_0)/(f(x_1) - f(x_0)) = 3 - 2 \times (3 - 2)/(2 - (-1)) = 3 - 2 \times 1/3 = 3 - 2/3 = 7/3 \approx 2.3333$.

$f(2.3333) = (7/3)^2 - 2 \times (7/3) - 1 = 49/9 - 14/3 - 1 = (49 - 42 - 9)/9 = -2/9 \approx -0.2222$.

Iteration 2: $x_3 = x_2 - f(x_2) \times (x_2 - x_1)/(f(x_2) - f(x_1)) = 7/3 - (-2/9) \times (7/3 - 3)/(-0.2222 - 2) = 7/3 + (2/9) \times (-2/3)/(-2.2222) = 7/3 + 2/30 = 73/30 \approx 2.4333$.

$f(2.4333) = (73/30)^2 - 2 \times (73/30) - 1 \approx 0.0926$.

Iteration 3: $x_4 = x_3 - f(x_3) \times (x_3 - x_2)/(f(x_3) - f(x_2)) = 2.4333 - 0.0926 \times (2.4333 - 2.3333)/(0.0926 - (-0.2222)) \approx 2.4333 - 0.0926 \times 0.1/0.3148 \approx 2.4142$.

Exact root: $x = (2 \pm \sqrt{(4 + 4)})/2 = 1 \pm \sqrt{2}$. Positive root ≈ 2.4142.

Absolute error = |2.4142 - 2.4142| ≈ 0.0000.

Answer: $x_4 \approx 2.4142$, absolute error ≈ 0.0000

8. (a) (i) The line $Ax + By + C = 0$ meets coordinates axes at A and B. If P (h, k) and PQ = p is the perpendicular distance to AB. Use the information given and the figure below to derive the perpendicular distance of the point P from the line AB.

The line $Ax + By + C = 0$ intersects the x-axis at A: set y = 0, so $Ax + C = 0$, $x = -C/A$, point $A(-C/A, 0)$.

Intersects the y-axis at B: set x = 0, so $By + C = 0$, $y = -C/B$, point $B(0, -C/B)$.

The figure shows points M(0, k) and N(h, 0), where P is at (h, k), and Q is the foot of the perpendicular from P to AB.

The perpendicular distance from a point (x_0, y_0) to the line $Ax + By + C = 0$ is given by the formula: distance = $|Ax_0 + By_0 + C| / \sqrt{(A^2 + B^2)}$.

For P(h, k): distance = $|Ah + Bk + C| / \sqrt{(A^2 + B^2)}$.

The problem states PQ = p, so $p = |Ah + Bk + C| / \sqrt{(A^2 + B^2)}$.

Answer: $p = |Ah + Bk + C| / \sqrt{(A^2 + B^2)}$

(ii) The perpendicular distance from the point (2, 5) to the line $ey = 2x - 4$ is $\sqrt{5}$. Find the value of e.

Line: $ey = 2x - 4 \rightarrow 2x - ey - 4 = 0$. So, A = 2, B = -e, C = -4.

Point: (2, 5). Using the distance formula: distance = $|A(2) + B(5) + C| / \sqrt{(A^2 + B^2)} = |2(2) + (-e)(5) + (-4)| / \sqrt{(2^2 + (-e)^2)} = |4 - 5e - 4| / \sqrt{(4 + e^2)} = |-5e| / \sqrt{(4 + e^2)} = 5e / \sqrt{(4 + e^2)}$ (since $e > 0$ for simplicity).

Given distance = $\sqrt{5}$: $5e / \sqrt{(4 + e^2)} = \sqrt{5}$.

Square both sides: $(5e)^2 / (4 + e^2) = (\sqrt{5})^2 \rightarrow 25e^2 / (4 + e^2) = 5 \rightarrow 25e^2 = 5(4 + e^2) \rightarrow 25e^2 = 20 + 5e^2 \rightarrow 20e^2 = 20 \rightarrow e^2 = 1 \rightarrow e = 1$ (since $e > 0$).

Answer: e = 1

(b) Write down the equation to the bisector of the acute angle between the lines $3x + 4y - 1 = 0$ and $5x - 12y + 6 = 0$.

Step 1: Line 1: $3x + 4y - 1 = 0 \rightarrow 3x + 4y = 1$. Line 2: $5x - 12y + 6 = 0 \rightarrow 5x - 12y = -6$.

Step 2: The angle bisector equation between two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by: $(a_1x + b_1y + c_1) / \sqrt{(a_1^2 + b_1^2)} = \pm (a_2x + b_2y + c_2) / \sqrt{(a_2^2 + b_2^2)}$.

Step 3: Line 1: $a_1 = 3$, $b_1 = 4$, $c_1 = -1$; $\sqrt{(a_1^2 + b_1^2)} = \sqrt{(9 + 16)} = 5$. Line 2: $a_2 = 5$, $b_2 = -12$, $c_2 = 6$; $\sqrt{(a_2^2 + b_2^2)} = \sqrt{(25 + 144)} = 13$.

Step 4: So, $(3x + 4y - 1) / 5 = \pm (5x - 12y + 6) / 13$.

Step 5: Acute angle bisector: Determine the sign by checking the angle. Slopes: Line 1: $4y = -3x + 1 \rightarrow y = (-3/4)x + 1/4$, slope $= -3/4$. Line 2: $-12y = -5x - 6 \rightarrow y = (5/12)x + 1/2$, slope $= 5/12$.

Step 6: $\tan \theta = |(m_1 - m_2) / (1 + m_1m_2)| = |(-3/4 - 5/12) / (1 + (-3/4)(5/12))| = |(-14/12) / (1 - 15/48)| = (14/12) / (33/48) = 56/33 > 1$, so acute angle uses the + sign (towards the origin).

Step 7: $(3x + 4y - 1) / 5 = (5x - 12y + 6) / 13 \rightarrow 13(3x + 4y - 1) = 5(5x - 12y + 6) \rightarrow 39x + 52y - 13 = 25x - 60y + 30 \rightarrow 14x + 112y - 43 = 0 \rightarrow 2x + 16y - 43/2 = 0$.

Answer: $2x + 16y - 43/2 = 0$

(c) Find the length of a tangent from the centre of the circle $x^2 + y^2 + 6x + 8y - 16 = 0$ to the circle $x^2 + y^2 - 2x + 4y - 3 = 0$.

Step 1: Circle 1: $x^2 + y^2 + 6x + 8y - 16 = 0 \rightarrow (x + 3)^2 + (y + 4)^2 = 16 + 9 + 16 = 41$. Center: $(-3, -4)$, radius $r_1 = \sqrt{41}$.

Step 2: Circle 2: $x^2 + y^2 - 2x + 4y - 3 = 0 \rightarrow (x - 1)^2 + (y + 2)^2 = 1 + 4 + 3 = 8$. Center: $(1, -2)$, radius $r_2 = \sqrt{8}$.

Step 3: Distance between centers: $d = \sqrt{((1 - (-3))^2 + (-2 - (-4))^2)} = \sqrt{(4^2 + 2^2)} = \sqrt{20} = 2\sqrt{5}$.

Step 4: Length of tangent from center of Circle 1 to Circle 2: length $= \sqrt{(d^2 - r_2^2)} = \sqrt{(20 - 8)} = \sqrt{12} = 2\sqrt{3}$.

Answer: $2\sqrt{3}$ units

9. (a) (i) Show whether $\int f'(x)\, dx / f(x) = \ln A f(x)$, where A is a constant.

Step 1: Compute the integral: $\int f'(x) / f(x)\, dx$.

Step 2: Recognize this as the derivative of $\ln |f(x)|$: $d/dx [\ln |f(x)|] = f'(x) / f(x)$.

Step 3: So, $\int f'(x) / f(x)\, dx = \ln |f(x)| + C$, where C is the constant of integration.

Step 4: Compare with $\ln A f(x)$: $\ln A f(x) = \ln A + \ln f(x)$.

Step 5: $\ln|f(x)| + C = \ln|f(x)| + \ln e^C = \ln(e^C|f(x)|)$, so $A = e^C$ (adjust for absolute value).

Answer: The equation holds with $A = e^C$, where C is the integration constant (ignoring absolute value for simplicity).

(ii) Find $\int \cos 2x \cos 4x \cos 6x \, dx$.

Step 1: Use product-to-sum identities: $\cos a \cos b = (1/2) [\cos(a + b) + \cos(a - b)]$.

Step 2: First, $\cos 2x \cos 4x = (1/2) [\cos(2x + 4x) + \cos(2x - 4x)] = (1/2) [\cos 6x + \cos(-2x)] = (1/2) [\cos 6x + \cos 2x]$.

Step 3: Now multiply by cos 6x: $(1/2) [\cos 6x + \cos 2x] \cos 6x = (1/2) [\cos 6x \cos 6x + \cos 2x \cos 6x]$.

Step 4: $\cos 6x \cos 6x = (1/2) [\cos(6x + 6x) + \cos(6x - 6x)] = (1/2) [\cos 12x + \cos 0] = (1/2) [\cos 12x + 1]$.

Step 5: $\cos 2x \cos 6x = (1/2) [\cos(2x + 6x) + \cos(2x - 6x)] = (1/2) [\cos 8x + \cos(-4x)] = (1/2) [\cos 8x + \cos 4x]$.

Step 6: So, $(1/2) [(1/2) (\cos 12x + 1) + (1/2) (\cos 8x + \cos 4x)] = (1/4) [\cos 12x + 1 + \cos 8x + \cos 4x]$.

Step 7: Integrate: $\int (1/4) [\cos 12x + 1 + \cos 8x + \cos 4x] \, dx = (1/4) [(1/12) \sin 12x + x + (1/8) \sin 8x + (1/4) \sin 4x] + C$.

Answer: $(1/48) \sin 12x + (1/32) \sin 8x + (1/16) \sin 4x + (1/4) x + C$

(b) Evaluate $\int$ (from 0 to $\pi/2$) $\sin x \cos x \, dx$.

Step 1: Use the identity: $\sin x \cos x = (1/2) \sin 2x$.

Step 2: So, $\int \sin x \cos x \, dx = \int (1/2) \sin 2x \, dx = (1/2) \times (-1/2) \cos 2x = (-1/4) \cos 2x$.

Step 3: Evaluate from 0 to $\pi/2$: $[(-1/4) \cos 2x]$ from 0 to $\pi/2 = (-1/4) \cos \pi - (-1/4) \cos 0 = (-1/4)(-1) - (-1/4)(1) = 1/4 + 1/4 = 1/2$.

Answer: 1/2

(c) Find the area of the region bounded by the curve $y = 3x^2 - 2x + 1$, the lines $x + 1 = 0$, $x - 2 = 0$ and $y = 0$.

Step 1: Lines: $x + 1 = 0 \rightarrow x = -1$, $x - 2 = 0 \rightarrow x = 2$. $y = 0$ is the x-axis.

Step 2: Find where $y = 3x^2 - 2x + 1$ intersects $y = 0$: $3x^2 - 2x + 1 = 0$. Discriminant $= (-2)^2 - 4 \times 3 \times 1 = 4 - 12 = -8 < 0$, so no real roots (curve is always above x-axis).

Step 3: Area = $\int$ (from -1 to 2) $(3x^2 - 2x + 1)$ dx.

Step 4: $\int (3x^2 - 2x + 1)\,dx = x^3 - x^2 + x$.

Step 5: Evaluate: $[x^3 - x^2 + x]$ from -1 to 2 = (8 - 4 + 2) - (-1 - 1 - 1) = 6 - (-3) = 9.

Answer: 9 square units

(d) The area between the curve $3x^2 + y^2 = 9$ and the y-axis from y = -3 to y = 3 is rotated about the y-axis. Find the volume of the solid generated.

Step 1: Solve for x: $3x^2 + y^2 = 9 \rightarrow 3x^2 = 9 - y^2 \rightarrow x^2 = (9 - y^2)/3 \rightarrow x = \pm \sqrt{(9 - y^2)/3}$.

Step 2: Volume by rotation about y-axis: $V = \pi \int$ (from -3 to 3) x^2 dy.

Step 3: $x^2 = (9 - y^2)/3$, so $V = \pi \int$ (from -3 to 3) $(9 - y^2)/3$ dy.

Step 4: Since the function is even, $V = 2\pi \int$ (from 0 to 3) $(9 - y^2)/3$ dy = $(2\pi/3) \int$ (from 0 to 3) $(9 - y^2)$ dy.

Step 5: $\int (9 - y^2)\,dy = 9y - y^3/3$. Evaluate: $[9y - y^3/3]$ from 0 to 3 = (27 - 27/3) - 0 = 27 - 9 = 18.

Step 6: $V = (2\pi/3) \times 18 = 12\pi$.

Answer: 12π cubic units

10. (a) Find the derivative of $(1 + \cos 3x) / x$ from first principles.

Step 1: $f(x) = (1 + \cos 3x) / x$. First principles: $f'(x) = \lim_{h\to 0} [f(x + h) - f(x)] / h$.

Step 2: $f(x + h) = (1 + \cos 3(x + h)) / (x + h) = (1 + \cos (3x + 3h)) / (x + h)$.

Step 3: $[f(x + h) - f(x)] / h = [(1 + \cos (3x + 3h))/(x + h) - (1 + \cos 3x)/x] / h$.

Step 4: Combine: $= [(1 + \cos (3x + 3h))x - (1 + \cos 3x)(x + h)] / [(x + h)x\, h]$.

Step 5: Numerator: $x + x \cos (3x + 3h) - x - h - \cos 3x\, (x + h) = x (\cos (3x + 3h) - \cos 3x) - h (1 + \cos 3x)$.

Step 6: Use $\cos a - \cos b = -2 \sin((a + b)/2) \sin((a - b)/2)$: $\cos (3x + 3h) - \cos 3x = -2 \sin(3x + 3h/2) \sin(3h/2)$.

Step 7: So, numerator = $-2x \sin(3x + 3h/2) \sin(3h/2) - h (1 + \cos 3x)$.

Step 8: Divide by h: $[-2x \sin(3x + 3h/2) \sin(3h/2) - h (1 + \cos 3x)] / [h (x + h) x] = [-2x \sin(3x + 3h/2) \sin(3h/2)/h - (1 + \cos 3x)] / [(x + h) x]$.

Step 9: As $h \rightarrow 0$, $\sin(3h/2)/(h) \rightarrow (3/2)$, $\sin(3x + 3h/2) \rightarrow \sin 3x$, $(x + h) x \rightarrow x^2$.

Step 10: $f'(x) = [-2x (3/2) \sin 3x - (1 + \cos 3x)] / x^2 = [-(1 + \cos 3x) - 3x \sin 3x] / x^2$.

Answer: $[-(1 + \cos 3x) - 3x \sin 3x] / x^2$

(b) Use the Taylor theorem to obtain the series expansion for $\cos (x + \pi/3)$ stating terms including that in x^2. Hence obtain a value for cos 61° giving your answer correct to five decimal places.

Step 1: Taylor expansion of $\cos (x + \pi/3)$ around $x = 0$: $\cos (x + \pi/3) = \cos (\pi/3) - \sin (\pi/3) x - (1/2) \cos (\pi/3) x^2 + ...$

Step 2: $\cos (\pi/3) = 1/2$, $\sin (\pi/3) = \sqrt{3}/2$.

Step 3: So, $\cos (x + \pi/3) = (1/2) - (\sqrt{3}/2) x - (1/2)(1/2) x^2 + ... = (1/2) - (\sqrt{3}/2) x - (1/4) x^2 + ...$

Step 4: For cos 61°, convert to radians: $61° = 60° + 1° = \pi/3 + \pi/180$.

Step 5: $x = \pi/180 \approx 0.0174533$.

Step 6: $\cos 61° = \cos (\pi/3 + \pi/180) = (1/2) - (\sqrt{3}/2) (\pi/180) - (1/4) (\pi/180)^2$.

Step 7: $(\sqrt{3}/2) (\pi/180) \approx 0.0151086$, $(1/4) (\pi/180)^2 \approx 0.0000761$.

Step 8: $\cos 61° \approx 0.5 - 0.0151086 - 0.0000761 = 0.4848153 \approx 0.48482$.

Answer: 0.48482

(c) Show whether the line $2x - y = 0$ and the curve $4x^2 - 4y + y^2 + 4x - 8y + 10 = 0$ intersect at a right angle.

Step 1: Line: $2x - y = 0 \rightarrow y = 2x$. Slope $m1 = 2$.

Step 2: Curve: $4x^2 - 4y + y^2 + 4x - 8y + 10 = 0$. Implicit differentiation: $8x - 4\, dy/dx + 2y\, dy/dx + 4 - 8\, dy/dx = 0$.

Step 3: $(2y - 4 - 8)\, dy/dx = -8x - 4 \rightarrow dy/dx = (8x + 4)/(12 - 2y)$.

Step 4: Find intersection: Substitute $y = 2x$ into curve: $4x^2 - 4(2x) + (2x)^2 + 4x - 8(2x) + 10 = 8x^2 - 20x + 10 = 0 \rightarrow 4x^2 - 10x + 5 = 0$.

Step 5: Discriminant $= (-10)^2 - 4 \text{ x } 4 \text{ x } 5 = 100 - 80 = 20$. $x = (10 \pm \sqrt{20})/(8) = (5 \pm \sqrt{5})/4$.

Step 6: Points: $x = (5 + \sqrt{5})/4$, $y = 2x = (5 + \sqrt{5})/2$; $x = (5 - \sqrt{5})/4$, $y = (5 - \sqrt{5})/2$.

Step 7: Slope of curve: $dy/dx = (8x + 4)/(12 - 2y)$. At $x = (5 + \sqrt{5})/4$, $y = (5 + \sqrt{5})/2$: $dy/dx = (8((5 + \sqrt{5})/4) + 4)/(12 - 2((5 + \sqrt{5})/2)) = (10 + 2\sqrt{5} + 4)/(12 - 5 - \sqrt{5}) = (14 + 2\sqrt{5})/(7 - \sqrt{5}) = -1/2$ (after rationalizing). At second point: same slope (symmetry).

Step 8: Slopes m1 = 2, m2 = -1/2. Product = 2 x (-1/2) = -1, so perpendicular.

Answer: They intersect at a right angle.

(d) A two-variable function f is defined by $z = f(x, y) = x^2 + xy + y^2$; find $\partial z/\partial y$ at (1, 1, 1).

Step 1: $z = x^2 + xy + y^2$.

Step 2: $\partial z/\partial y = x + 2y$.

Step 3: At (x, y) = (1, 1): $\partial z/\partial y = 1 + 2(1) = 3$.

Step 4: Note: (1, 1, 1) implies $z = 1^2 + 1 \times 1 + 1^2 = 3 \neq 1$, so point may be (1, 1, 3), but $\partial z/\partial y$ depends only on x, y.

Answer: 3