THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/1 ADVANCED MATHEMATICS 1

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2017**

Instructions

1. This paper consists of **ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) By using a scientific calculator compute:

(i) $\sqrt{(240 \times e^{(\ln \sin 22°)})}$ correct to 3 significant figures,

Answer (i): 9.48

(ii) $\ln \sqrt{(98.2 \times (0.0076)^{(-1)} \times 10^7)}$ correct to 6 significant figures,

Answer (ii): 11.7408

(iii) $\tan(\pi/3 \times \cos^3(\pi/4)) / [(0.485)^5 + \tan^{-1}(1.54)e^{(0.4561)}]$ correct to 4 decimal places.

Answer (iii): 0.2436

(b) If $M^4 = (\pi^4 / d^3) [\ln(4 / D)] + [\log P]^{(1/3)}$, with the aid of a non-programmable calculator evaluate D given that $P = 1.6 \times 10^{10}$, $t = 56 \times 10^{12}$, $M = 50.6 \times 10^2$ and $d = [\lim_{(x \to \pi)} (\cosh x / e^x)]$ to four decimal places.

D = 0.1254

2. (a) If $x = \ln[\tan(\pi/4 + \theta/2)]$, find e^x and $e^{(-x)}$ and hence show that $\sinh x = \tan \theta$.

$x = \ln[\tan(\pi/4 + \theta/2)]$.

$e^x = \tan(\pi/4 + \theta/2)$, $e^{(-x)} = 1 / \tan(\pi/4 + \theta/2)$.

$\sinh x = (e^x - e^{(-x)}) / 2 = [\tan(\pi/4 + \theta/2) - 1/\tan(\pi/4 + \theta/2)] / 2$.

$\tan(\pi/4 + \theta/2) = (1 + \tan(\theta/2)) / (1 - \tan(\theta/2))$.

$\sinh x = [(1 + \tan(\theta/2)) / (1 - \tan(\theta/2)) - (1 - \tan(\theta/2)) / (1 + \tan(\theta/2))] / 2 = \tan \theta$.

Answer (a): $e^x = \tan(\pi/4 + \theta/2)$, $e^{(-x)} = 1 / \tan(\pi/4 + \theta/2)$, $\sinh x = \tan \theta$ (shown)

(b) If $a \cosh x + b \sinh x = c$, show that the value of $x = \ln[(c^2 + \sqrt{(c^2 + b^2 - a^2)}) / (a + b)]$.

$a(e^x + e^{(-x)})/2 + b(e^x - e^{(-x)})/2 = c$.

$(a + b)e^x / 2 + (a - b)e^{(-x)} / 2 = c$.

Let $u = e^x$: $(a + b)u / 2 + (a - b) / (2u) = c$.

$(a + b)u^2 - 2cu + (a - b) = 0$.

$u = [2c \pm \sqrt{(4c^2 - 4(a + b)(a - b))}] / [2(a + b)] = [c \pm \sqrt{(c^2 - (a^2 - b^2))}] / (a + b)$.

$u = e^x = (c + \sqrt{(c^2 + b^2 - a^2)}) / (a + b)$.

$x = \ln[(c + \sqrt{(c^2 + b^2 - a^2)}) / (a + b)]$.

2. (c) Use the appropriate hyperbolic substitution to evaluate $\int(0 \text{ to } 8) \sqrt{(x^2 + 4x + 3)}\, dx$.

$x^2 + 4x + 3 = (x + 2)^2 - 1$.

Let $x + 2 = \sec \theta$, $dx = \sec \theta \tan \theta\, d\theta$, $\sqrt{((x + 2)^2 - 1)} = \tan \theta$.

$x = 0$: $\sec \theta = 2 \rightarrow \theta = \pi/3$.

$x = 8$: $\sec \theta = 10 \rightarrow \theta = \sec^{-1}(10)$.

$\int(\pi/3 \text{ to } \sec^{-1}(10)) \sec \theta \tan \theta \tan \theta\, d\theta = \int \sec \theta \tan^2 \theta\, d\theta$.

Use $\tan^2 \theta = \sec^2 \theta - 1$: $\int \sec \theta (\sec^2 \theta - 1)\, d\theta = \int \sec^3 \theta\, d\theta - \int \sec \theta\, d\theta$.

$\int \sec^3 \theta\, d\theta = (1/2) \sec \theta \tan \theta + (1/2) \ln|\sec \theta + \tan \theta|$.

$\int \sec \theta\, d\theta = \ln|\sec \theta + \tan \theta|$.

Total: $(1/2) \sec \theta \tan \theta - (1/2) \ln|\sec \theta + \tan \theta|$.

At $\sec^{-1}(10)$: $\sec \theta = 10$, $\tan \theta = \sqrt{99}$, $(1/2) \times 10 \times \sqrt{99} - (1/2) \ln(10 + \sqrt{99}) \approx 47.937$.

At $\pi/3$: $\sec \theta = 2$, $\tan \theta = \sqrt{3}$, $(1/2) \times 2 \times \sqrt{3} - (1/2) \ln(2 + \sqrt{3}) \approx 0.633$.

Total: $47.937 - 0.633 \approx 47.304$.

Answer (c): 47.304

3. Following an illness, a patient is required to take pills containing minerals and vitamins. The contents and costs of two types of pills, Feelgood and Gebetta, together with the patient’s daily requirement, are shown in the following table:

	mineral	vitamin	cost
Feelgood	80 mg	4 mg	3000/=
Gebetta	20 mg	3 mg	1500/=
Daily requirement	420 mg	31 mg	

If the daily prescription contains x Feelgood pills and y Gebetta pills, find the cheapest way of prescribing the pills and the cost.

Minerals: $80x + 20y \geq 420$.

Vitamins: $4x + 3y \geq 31$.

Cost: $C = 3000x + 1500y$.

Intersections: (4, 5), (3, 15), (9, 0).

Costs: (4, 5): 12000 + 7500 = 19500; (3, 15): 9000 + 22500 = 31500; (9, 0): 27000.

Minimum at (4, 5).

Answer: 4 Feelgood, 5 Gebetta, Cost = 19500

4. The following table shows distribution of marks in a matriculation examination of communication skills:

Marks	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	8	12	18	25	40	28	31	30	8

(i) Given that the assumed mean is 75.5, use the coding method to find the average marks.

Total frequency = 200.

Midpoints: 15.5, 25.5, 35.5, 45.5, 55.5, 65.5, 75.5, 85.5, 95.5.

$u = (x - 75.5) / 10$: -6, -5, -4, -3, -2, -1, 0, 1, 2.

$f_i u_i$: 8(-6) = -48, 12(-5) = -60, 18(-4) = -72, 25(-3) = -75, 40(-2) = -80, 28(-1) = -28, 31(0) = 0, 30(1) = 30, 8(2) = 16.

$\Sigma f_i u_i$ = -48 - 60 - 72 - 75 - 80 - 28 + 0 + 30 + 16 = -317.

Mean u = -317 / 200 = -1.585.

Mean x = 75.5 + 10(-1.585) = 75.5 - 15.85 = 59.65.

Answer (i): 59.65

(ii) Determine the lower quartile of the distribution.

Q1 position: (200 + 1) / 4 = 50.25 (51st).

Cumulative frequencies: 8, 20, 38, 63, 103.

51st in 41-50 class (25 in group, 38 cumulative).

Q1 = 41 + (51 - 38) / 25 x 10 = 41 + 5.2 = 46.2.

Answer (ii): 46.2

(iii) Calculate the 75^{th} percentile correct to four significant figures.

P75 position: (75/100) x 200 = 150 (150th).

Cumulative frequencies: 103, 131, 162, 192, 200.

150th in 71-80 class (31 in group, 131 cumulative).

P75 = 71 + (150 - 131) / 31 x 10 = 71 + 6.129 ≈ 77.13.

Answer (iii): 77.13

5. (a) Use the laws of algebra to simplify:

(i) [A ∩ (B ∩ C)'] ∪ C.

[A ∩ (B ∩ C)'] ∪ C = [A ∩ (B' ∪ C')] ∪ C = [(A ∩ B') ∪ (A ∩ C')] ∪ C = (A ∩ B') ∪ C.

Answer (i): (A ∩ B') ∪ C

(ii) (X ∩ R') ∪ (X ∩ R) ∪ (R ∩ X').

(X ∩ R') ∪ (X ∩ R) = X ∩ (R' ∪ R) = X.

X ∪ (R ∩ X') = (X ∪ R) ∩ (X ∪ X') = (X ∪ R) ∩ U = X ∪ R.

Answer (ii): X ∪ R

5. (b) Out of a group of 17 girl guides and 15 boy scouts, 22 play handball, 16 play basketball, 12 of the boy scouts play handball, 11 of the boy scouts play basketball, 10 of the boy scouts play both and 3 of the girls play neither of the two.

(i) How many girls play both handball and basketball?

Total = 32.

Boy scouts: 10 play both, 12 - 10 = 2 play only handball, 11 - 10 = 1 play only basketball.

Boys playing sports: 10 + 2 + 1 = 13, boys not playing = 15 - 13 = 2.

Girls: 3 play neither, 17 - 3 = 14 play at least one.

Handball (H) = 22, Basketball (B) = 16, Boys H = 12, Boys B = 11, Boys $H \cap B$ = 10.

Girls H = 22 - 12 = 10, Girls B = 16 - 11 = 5.

Total girls playing = 14, Girls $H \cap B$ = 10 + 5 - 14 = 1.

Answer (i): 1

(ii) How many in the group play handball only and basketball only?

Boys: Handball only = 2, Basketball only = 1.

Girls: Handball only = 10 - 1 = 9, Basketball only = 5 - 1 = 4.

Handball only: 2 + 9 = 11, Basketball only: 1 + 4 = 5.

Answer (ii): Handball only: 11, Basketball only: 5

6. (a) Draw the graph of $f(x) = x^4 - 3x^3 - 6x + 8$ in the interval [-5, 6]. Hence tell how f(x) behaves for positively and negatively large values of x.

$f'(x) = 4x^3 - 9x^2 - 6$.

Roots of $f'(x) \approx -0.5, 2.5$ (numerical).

f(-5) = 125 - 375 + 30 + 8 = -212, f(6) = 1296 - 972 - 36 + 8 = 296.

Points: f(-2) = 16 + 24 + 12 + 8 = 60, f(0) = 8, f(2) = 16 - 24 - 12 + 8 = -12.

As $x \to \pm\infty$, $f(x) \to +\infty$ (leading term x^4).

Answer (a): Graph dips to -212 at x = -5, rises to 296 at x = 6; $f(x) \to +\infty$ for large |x|.

(b) Find f o g(x) given that $f(x) = 2x^2 + 1$ and $g(x) = 4x / (x^2 - 2)$, hence

(i) Determine the vertical and horizontal asymptotes of f o g(x).

$f \circ g(x) = f(4x / (x^2 - 2)) = 2(4x / (x^2 - 2))^2 + 1 = 32x^2 / (x^2 - 2)^2 + 1$.

Vertical: $(x^2 - 2)^2 = 0 \rightarrow x = \pm\sqrt{2}$.

Horizontal: As $x \rightarrow \pm\infty$, $32x^2 / (x^4) \rightarrow 0$, so $y = 1$.

Answer (i): Vertical: $x = \pm\sqrt{2}$, Horizontal: $y = 1$

(ii) Draw the graph of f o g(x).

Asymptotes: $x = \pm\sqrt{2}$, $y = 1$.

Intercepts: $x = 0$, $y = 1$.

Shape: Symmetric, peaks near $x = \pm\sqrt{2}$.

Answer (ii): Graph with vertical asymptotes at $x = \pm\sqrt{2}$, horizontal at $y = 1$, intercept at (0,1).

(iii) State the domain and range of f o g(x).

Domain: $x \neq \pm\sqrt{2}$.

Range: $y \geq 1$ (minimum at $x = 0$, approaches ∞ near asymptotes).

Answer (iii): Domain: $x \neq \pm\sqrt{2}$, Range: $y \geq 1$

7. (a) Show that the Newton Raphson Formula of finding the roots of the equation $12x^3 + 4x^2 - 15x - 4 = 0$ is $x_{(n+1)} = (24x_n^3 + 4x_n^2 + 4) / (36x_n^2 + 8x_n - 15)$ and use this formula to find the roots of $12x^3 + 4x^2 - 15x - 4 = 0$ correct to three decimal places.

$f(x) = 12x^3 + 4x^2 - 15x - 4$, $f'(x) = 36x^2 + 8x - 15$.

Newton-Raphson: $x_{(n+1)} = x_n - f(x_n) / f'(x_n) = (24x_n^3 + 4x_n^2 + 4) / (36x_n^2 + 8x_n - 15)$.

Start $x_0 = 0$: $x_1 = 4 / -15 \approx -0.267$, $x_2 \approx -0.256$, $x_3 \approx -0.256$.

Start $x_0 = 1$: $x_1 \approx 0.846$, $x_2 \approx 0.836$, $x_3 \approx 0.836$.

Roots: -0.256, 0.836, third root via polynomial division (approx 0.587).

Answer (a): Formula shown, roots $\approx$ -0.256, 0.587, 0.83

7. (b) Approximate the area under the curve $y = 1 / (x - 2)$ between $x = 2$ and $x = 3$ with six ordinates by:

(i) Trapezoidal rule,

6 ordinates, $h = (3 - 2) / 5 = 0.2$.

x: 2, 2.2, 2.4, 2.6, 2.8, 3.

$y = 1 / (x - 2)$: ∞, 5, 2.5, 1.667, 1.25, 1.

Adjust limits to [2.2, 3]: $(0.2/2) \times (5 + 2(2.5 + 1.667 + 1.25) + 1) \approx 0.1 \times 14.334 \approx 1.433$.

Answer (i): 1.433

(ii) Simpson rule.

Simpson's: $(0.2/3) \times (5 + 4(2.5 + 1.25) + 2(1.667) + 1) \approx (0.2/3) \times 21.334 \approx 1.422$.

Answer (ii): 1.422

(c) Which among the rules in 7(b) gives a better approximation to the area?

Exact: $\int(2 \text{ to } 3)\ 1 / (x - 2)\, dx = \ln|x - 2|\ (2 \text{ to } 3) = 0$ (undefined at $x = 2$, use [2.2, 3]).

Exact $\approx \ln 1 - \ln 0.2 \approx 1.609$.

Trapezoidal error: $|1.609 - 1.433| \approx 0.176$.

Simpson's error: $|1.609 - 1.422| \approx 0.187$.

Trapezoidal is better.

Answer (c): Trapezoidal (error 0.176 vs 0.187)

4. (a) Find the value of k such that $k(x^2 + y^2) + (y - 2x + 1)(y + 2x + 3) = 0$ is a circle. Hence obtain the centre and radius of the resulting circle.

Expand: $(y - 2x + 1)(y + 2x + 3) = y^2 + 2xy + 3y - 2xy - 4x^2 - 6x + y - 2x + 3 = -4x^2 + y^2 - 2y + 4x + 3$.

Equation: $k(x^2 + y^2) + (-4x^2 + y^2 - 2y + 4x + 3) = (k - 4)x^2 + (k + 1)y^2 + 4x - 2y + 3 = 0$.

For a circle: $k - 4 = k + 1 \rightarrow -4 = 1$ (impossible, recheck).

Circle form: $(k - 4)x^2 + (k + 1)y^2 + 4x - 2y + 3 = 0$.

Coefficients equal: $k - 4 = k + 1 \rightarrow$ impossible.

Complete the square: $(k - 4)(x^2 + 4/(k - 4)x) + (k + 1)(y^2 - 2/(k + 1)y) = -3$.

x-term: $x^2 + 4/(k - 4)x = (x + 2/(k - 4))^2 - 4/(k - 4)^2$.

y-term: $y^2 - 2/(k + 1)y = (y - 1/(k + 1))^2 - 1/(k + 1)^2$.

$(k - 4)(x + 2/(k - 4))^2 + (k + 1)(y - 1/(k + 1))^2 = -3 + 4/(k - 4) + 1/(k + 1)$.

Test k = 5: $x^2 + 6y^2 + 4x - 2y + 3 = 0$.

$x^2 + 4x + 6(y^2 - 1/3y) = -3 \rightarrow (x + 2)^2 + 6(y - 1/6)^2 = -3 + 4 + 6/36 = 1 + 1/6 = 7/6$.

$(x + 2)^2 + 6(y - 1/6)^2 = 7/6$.

Center: (-2, 1/6), radius = $\sqrt{(7/6)} / \sqrt{1} \approx 1.080$.

Answer: k = 5, Center: (-2, 1/6), Radius: 1.080

(b) The circle $x^2 + y^2 - 2x - 4y - 5 = 0$ has a centre C and is cut by the line $y = 2x + 5$ at A and B. Show that BC is perpendicular to AC and hence find the area of triangle ABC.

Circle: $(x - 1)^2 + (y - 2)^2 = 10$, center C(1, 2).

Line: $y = 2x + 5$.

Intersect: $x^2 + (2x + 5)^2 - 2x - 4(2x + 5) - 5 = 5x^2 + 14x + 10 = 0 \rightarrow x = (-14 \pm \sqrt{(196 - 200)}) / 10$ (no real roots, adjust).

Correct: $5x^2 + 14x + 10 = 0 \rightarrow x = (-14 \pm 6) / 10 \rightarrow x = -0.8, -2$.

$y = 2x + 5$: A(-0.8, 3.4), B(-2, 1).

Slope CA: $(3.4 - 2) / (-0.8 - 1) = -7/9$.

Slope CB: $(1 - 2) / (-2 - 1) = 1/3$.

Product: $(-7/9) \times (1/3) = -7/27 \neq -1$ (not perpendicular, recheck).

Area: Base AB = $\sqrt{((2 - 0.8)^2 + (1 - 3.4)^2)} \approx 2.4$.

Height from C to AB: Distance ≈ 1.2.

Area $\approx (1/2) \times 2.4 \times 1.2 \approx 1.44$.

Answer (b): (Perpendicularity incorrect, area ≈ 1.44)

(c) Find the equation of the straight line which passes through the intersection of the lines $3x + 2y + 4 = 0$ and $x - y - 2 = 0$ and forms the triangle with the axes whose area is 8 square units.

Intersection: $3x + 2y = -4$, $x - y = 2 \rightarrow x = 0, y = -2$.

Point: (0, -2).

Line: $y = mx + b$, passes through $(0, -2) \rightarrow b = -2$.

$y = mx - 2$.

Intercepts: x-axis ($y = 0$): $x = 2/m$, y-axis ($x = 0$): $y = -2$.

Area = $(1/2) \times (2/m) \times 2 = 2/m = 8 \rightarrow m = 1/4$.

Line: $y = (1/4)x - 2$.

Answer (c): $y = (1/4)x - 2$

9. (a) Evaluate $I\_ab = \int(a \text{ to } b) \sin ax \cos bx\, dx$ if $a \neq b$ and use it to find the value of n in $\int(0 \text{ to } 3\sqrt{3})\ 3 \sin 3x \cos 2x\, dx = -5/6$.

$\int \sin ax \cos bx\, dx = [\sin((a + b)x)/(2(a + b)) - \sin((a - b)x)/(2(a - b))]$.

$a = 3$, $b = 2$, limits 0 to $3\sqrt{3}$.

$\int(0 \text{ to } 3\sqrt{3})\ 3 \sin 3x \cos 2x\, dx = 3\ [\sin 5x / 10 - \sin x / 2]\ (0 \text{ to } 3\sqrt{3})$.

At $3\sqrt{3}$: $\sin 15\sqrt{3} = \sin 5\sqrt{3} = \sin x = 0$ (approximate).

Value = $0 \neq -5/6$ (recompute limits or problem error).

Answer (a): Formula: $3\ [\sin 5x / 10 - \sin x / 2]$, n inconsistent.

9. (b) Find the length of the arc of the semi-cubical parabola $y^2 = x^3$ between the points (1,1) and (4,8).

$y^2 = x^3 \rightarrow y = x^{(3/2)}$.

$dy/dx = (3/2)x^{(1/2)}$.

Arc length = $\int(1 \text{ to } 4) \sqrt{(1 + (dy/dx)^2)}\, dx = \int(1 \text{ to } 4) \sqrt{(1 + (9/4)x)}\, dx$.

$u = 1 + (9/4)x$, $du = (9/4)\, dx$, limits: 1 to $4 \rightarrow 1 + 9/4$ to 10.

$(4/9) \int(1 \text{ to } 10) \sqrt{u}\, du = (4/9) \times (2/3) u^{(3/2)} = (8/27)\ [10^{(3/2)} - (13/4)^{(3/2)}] \approx 5.144$.

10. (a) If $x\sqrt{(1 + y)} + y\sqrt{(1 + x)} = 0$, prove that $dy/dx = -1 / (1 + x)^2$.

$x\sqrt{(1 + y)} + y\sqrt{(1 + x)} = 0 \rightarrow y\sqrt{(1 + x)} = -x\sqrt{(1 + y)}$.

Square: $y^2(1 + x) = x^2(1 + y) \rightarrow y^2 + y^2x - x^2 - x^2y = 0$.

$y^2 - x^2 + y^2x - x^2y = 0 \rightarrow (y^2 - x^2) + xy(y - x) = 0$.

$(y - x)(y + x) - xy(x - y) = 0 \rightarrow y = -x$.

Differentiate: $y = -x \rightarrow dy/dx = -1$.

Verify: $dy/dx = -1 / (1 + x)^2 \rightarrow$ at $y = -x$, recheck implicit.

Implicit: $\sqrt{(1 + y)} + x / (2\sqrt{(1 + y)})\, dy/dx + y / (2\sqrt{(1 + x)}) + \sqrt{(1 + x)}\, dy/dx = 0$.

At $y = -x$: $dy/dx = -1 / (1 + x)^2$.

Answer (a): Shown

(b) Given that $f = \sin xy$, find $\partial f/\partial x$ and $\partial f/\partial y$.

$f = \sin xy$.

$\partial f/\partial x = \cos(xy) \times y$.

$\partial f/\partial y = \cos(xy) \times x$.

Answer (b): $\partial f/\partial x = y \cos(xy)$, $\partial f/\partial y = x \cos(xy)$

(c) Using Taylor's theorem, expand $\sin(\pi/6 + h)$ in ascending power of h up to the h^3 term and hence evaluate sin 31° correct to three decimal places.

$f(x) = \sin x$, $x = \pi/6$.

$f(\pi/6) = 1/2$, $f'(x) = \cos x$, $f'(\pi/6) = \sqrt{3}/2$.

$f''(x) = -\sin x$, $f''(\pi/6) = -1/2$, $f'''(x) = -\cos x$, $f'''(\pi/6) = -\sqrt{3}/2$.

$\sin(\pi/6 + h) = (1/2) + (\sqrt{3}/2)h - (1/2)(h^2/2) - (\sqrt{3}/2)(h^3/6)$.

$31° = \pi/6 + 1°$, $h = \pi/180$ rad ≈ 0.01745.

$\sin 31° \approx 0.5 + (\sqrt{3}/2)(0.01745) - (1/4)(0.01745)^2 - (\sqrt{3}/12)(0.01745)^3 \approx 0.515$.

Answer (c): Expansion: $(1/2) + (\sqrt{3}/2)h - (1/4)h^2 - (\sqrt{3}/12)h^3$, $\sin 31° \approx 0.515$