**THE UNITED REPUBLIC OF TANZANIA**

**NATIONAL EXAMINATIONS COUNCIL**

**ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION**

**142/1 ADVANCED MATHEMATICS 1**

(For Both School and Private Candidates)

**Time: 3 Hours** **ANSWERS** **Year: 2019**

**Instructions**

1. This paper consists of **ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.



***Prepared by: Maria Marco for TETEA***

1. (a) By using a non-programmable calculator:

(i) Calculate $\log(e^{(e + 2\ln 5)} + \log 5)$. Give your answer correct to six decimal places.

Answer (i): 2.578639

(ii) Obtain the value of $\sqrt{[4.03]^3 \times [814765]^{35} / 5}$ correct to three significant figures.

Answer (ii): $2.68 \times 10^{98}$

(b) The monthly salaries in Tanzanian shillings for 20 employees of KNCU are 260,000.00, 170,000.00, 85,000.00, 505,000.00, 129,000.00, 89,000.00, 220,000.00, 600,000.00, 340,000.00, 144,000.00, 157,000.00, 103,000.00, 480,000.00, 790,000.00, 600,000.00, 219,000.00, 195,000.00, 128,000.00, 90,000.00, 102,000.00, 185,000.00. Use the statistical functions of the scientific calculator to calculate:

(i) the mean ($\bar{x}$) and

Answer (i): 269550

(ii) the standard deviation ($x\sigma_n$).

Answer (ii): 141550

2. (a) Solve the equation $\text{cosech}^{-1}(x) + \ln x - \ln 3 = 0$.

$\text{cosech}^{-1}(x) = \ln(1/x + \sqrt{1/x^2 + 1})$.

$\ln(1/x + \sqrt{1/x^2 + 1}) + \ln x - \ln 3 = 0$.

$\ln((1/x + \sqrt{1/x^2 + 1})\, x) = \ln 3$.

$(1 + \sqrt{1 + x^2}) / x = 3$.

$1 + \sqrt{1 + x^2} = 3x$.

$\sqrt{1 + x^2} = 3x - 1$.

$1 + x^2 = (3x - 1)^2$.

$x^2 = 9x^2 - 6x + 1$.

$8x^2 - 6x - 1 = 0$.

$x = (6 \pm \sqrt{(36 + 32)}) / 16 = (6 \pm \sqrt{68}) / 16 = (3 \pm \sqrt{17}) / 8$.

$x > 0$: $x = (3 + \sqrt{17}) / 8$.

Answer (a): $x = (3 + \sqrt{17}) / 8$

(b) Given that $\sinh x = \tan \theta$, prove that $x = \ln(\sec \theta + \tan \theta)$.

$\sinh x = (e^x - e^{-x}) / 2 = \tan \theta$.

$e^x - e^{-x} = 2\tan \theta$.

Let $u = e^x$: $u - 1/u = 2\tan \theta$.

$u^2 - 2u\tan \theta - 1 = 0$.

$u = (2\tan \theta \pm \sqrt{(4\tan^2\theta + 4)}) / 2 = \tan \theta + \sqrt{(\tan^2\theta + 1)} = \tan \theta + \sec \theta$.

$e^x = \sec \theta + \tan \theta \rightarrow x = \ln(\sec \theta + \tan \theta)$.

(c) Use the hyperbolic functions substitution to find $\int(0 \text{ to } 1)\ 1 / \sqrt{(x^2 + 8x + 17)}\ dx$.

$x^2 + 8x + 17 = (x + 4)^2 + 1$.

Let $x + 4 = \sinh u$, $dx = \cosh u\ du$, $\sqrt{((x + 4)^2 + 1)} = \sqrt{(\sinh^2 u + 1)} = \cosh u$.

$x = 0$: $\sinh u = -4$, $u = -\ln(4 + \sqrt{17})$.

$x = 1$: $\sinh u = -3$, $u = -\ln(3 + 2\sqrt{2})$.

$\int \cosh u / \cosh u\ du = \int du = u$.

From $u = -\ln(4 + \sqrt{17})$ to $-\ln(3 + 2\sqrt{2}) = \ln(4 + \sqrt{17}) - \ln(3 + 2\sqrt{2})$.

Answer (c): $\ln(4 + \sqrt{17}) - \ln(3 + 2\sqrt{2})$

3. Mr. Masumbuko has two traditional stores A and B for storing groundnuts. He stored 80 bags in A and 70 bags in B. Two customers C and D placed orders for 35 and 60 bags respectively. The transport costs per bag from each store are summarized in the following table:

| From | To C | To D |
|---|---|---|
| A | 8 | 12 |
| B | 10 | 13 |

(a) How many bags of groundnuts should the farmer deliver to each customer in order to minimize the transportation cost?

Variables: x (A to C), 35-x (A to D), y (B to C), 60-y (B to D).

A: $x + (35-x) = 35$.

B: $y + (60-y) = 60$.

Cost: $8x + 12(35-x) + 10y + 13(60-y) = -4x - 3y + 1200$.

C: $x + y = 35 \rightarrow y = 35-x$.

Cost = $-4x - 3(35-x) + 1200 = -x + 1095$.

Minimize: x max ($0 \leq x \leq 35$), cost = 1060 ($x = 35$).

$x = 35$, $y = 0$ → A to C: 35, A to D: 0, B to C: 0, B to D: 60.

Answer (a): A → C: 35, D: 0; B → C: 0, D: 60

(b) Determine the minimum transportation cost.

Cost = $8 \times 35 + 12 \times 0 + 10 \times 0 + 13 \times 60 = 280 + 780 = 1060$.

Answer (b): 1060

4. (a) The sum of 20 numbers is 320 and the sum of the squares of the numbers is 5840.

(i) Calculate the mean and standard deviation of the 20 numbers.

Mean = $320 / 20 = 16$.

Variance = $(\Sigma x_i^2 / n) - (\bar{x})^2 = 5840 / 20 - 16^2 = 292 - 256 = 36$.

Standard deviation = $\sqrt{36} = 6$.

Answer (i): Mean = 16, Standard deviation = 6

(ii) If one number is added to the 20 numbers so that the mean is unchanged, find this number and show whether the standard deviation will change or not.

New mean = 16, total = 320 + x = 21 x 16 → x = 16.

New $\Sigma x_i^2 = 5840 + 16^2 = 5840 + 256 = 6096$.

New variance = $6096 / 21 - 16^2 \approx 290.2857 - 256 = 34.2857$.

New standard deviation $\approx \sqrt{34.2857} \approx 5.85 < 6$ (changed).

Answer (ii): Number = 16, standard deviation changes (6 to 5.85).

4. (b) A watchman at Mimani city shopping centre recorded the length of time to the nearest minute that a sample of 131 cars was parked in their car park. The results were as follow:

| Time (minutes) | 5 – 10 | 11 – 16 | 17 – 22 | 23 – 28 | 29 – 34 | 35 - 40 |
|---|---|---|---|---|---|---|
| Frequency | 15 | 28 | 37 | 26 | 18 | 7 |

(i) Calculate the median time correct to four significant figures.

Total = 131, median position = 131 / 2 = 65.5 (66th).

Cumulative frequencies: 15, 43, 80, 106, 124, 131.

66th in 17-22 class.

Median = 17 + (66 - 43) / 37 x 5 = 17 + 3.108 = 20.108 ≈ 20.11.

Answer (i): 20.11

(ii) By using the coding method and the assumed mean A = 19, calculate the mean in two decimal places.

Midpoints: 7.5, 13.5, 19.5, 25.5, 31.5, 37.5.

$u = (x - 19) / 5$: -2.3, -1.1, 0.1, 1.3, 2.5, 3.7.

$f_i u_i$: 15(-2.3) = -34.5, 28(-1.1) = -30.8, 37(0.1) = 3.7, 26(1.3) = 33.8, 18(2.5) = 45, 7(3.7) = 25.9.

$\Sigma f_i u_i = -34.5 - 30.8 + 3.7 + 33.8 + 45 + 25.9 = 43.1$.

Mean u = 43.1 / 131 ≈ 0.329.

Mean x = 19 + 5 x 0.329 = 19 + 1.645 = 20.65.

Answer (ii): 20.65

5. (a) Use set properties to prove that for any non-empty sets A and B, $(A \cap B') \cup (B \cap A') = (A \cup B) - (A \cap B)'$.

$(A \cap B') \cup (B \cap A')$ = symmetric difference $(A \oplus B)$.

$(A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)' = (A \cup B) \cap (A' \cup B') = (A \cap B') \cup (B \cap A')$.

Equal.

5. (b) A student at the Sokoine University of Agriculture made a study about the types of livestock in a nearby village. The student came up with the following findings: 110 villagers kept goats, 73 villagers kept pigs; 59 villagers kept cattle and goats, 53 kept goats and pigs; 32 kept cattle and pigs; 20 villagers kept all three types of livestock. If the village has 200 occupants, by using Venn diagram, find the number of villagers who kept;

(i) only one type of livestock,

All three $(G \cap P \cap C) = 20$.

Goats and pigs only: 53 - 20 = 33.

Goats and cattle only: 59 - 20 = 39.

Pigs and cattle only: 32 - 20 = 12.

Goats only: 110 - (33 + 39 + 20) = 18.

Pigs only: 73 - (33 + 12 + 20) = 8.

Cattle only: (59 + 32 - 20 - 39 - 12 + 20) - 39 - 12 = 0.

Only one type: 18 + 8 + 0 = 26.

Answer (i): 26

(ii) only two types of livestock,

Only two: 39 + 33 + 12 = 84.

Answer (ii): 84

(iii) none of the livestock.

Total in diagram: 18 + 8 + 0 + 39 + 33 + 12 + 20 = 130.

None: 200 - 130 = 70.

Answer (iii): 70

6. (a) (i) Mention any two properties of f(x) = b^x.

b^x > 0 for all x.

b^x is increasing if b > 1.

Answer (i): 1. b^x > 0, 2. increasing if b > 1

(ii) Draw the graph of f(x) = (1/2) b^x for $-3 \leq x \leq 3$.

f(x) = (1/2) x 2^x.

x = -3: (1/2) x $2^{-3}$ = 1/16.

x = -2: 1/8.

x = -1: 1/4.

x = 0: 1/2.

x = 1: 1.

x = 2: 2.

x = 3: 4.

Exponential decay, y-intercept (0, 1/2).

Answer (ii): Exponential decay, points: (-3, 1/16), (0, 1/2), (3, 4).

(b) Given that y = ($x^2$ - 2x - 3) / ($x^2$ - 4)

(i) Find the vertical and horizontal asymptotes.

Vertical: $x^2 - 4 = 0 \rightarrow x = \pm 2$.

Horizontal: $y \approx 1$ (as $x \rightarrow \pm\infty$).

Answer (i): Vertical: $x = \pm 2$, Horizontal: y = 1

(ii) Sketch the graph of y.

Asymptotes: $x = \pm 2$, y = 1.

Intercepts: $y = 0$ at $x = 3$, $x = -1$; $x = 0$, $y = 3/4$.

Shape: Crosses at (-1,0), (3,0), approaches $y = 1$.

Answer (ii): Graph with vertical asymptotes at $x = \pm 2$, horizontal at $y = 1$, intercepts at (-1,0), (3,0), (0,3/4).

7. (a) The value $A = \int$(a to b) f(x) dx represents the area under the graph of $y = f(x)$ between $x = a$ and $x = b$. Derive the trapezium rule with 6 ordinates to find an approximation of $A = \int$(a to b) f(x) dx.

6 ordinates → 5 subintervals, $h = (b - a) / 5$.

Points: $x_0 = a$, $x_1 = a + h$, ..., $x_5 = b$.

Trapezium rule: $(h/2) \times (y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5)$.

Answer (a): $(h/2) \times (y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5)$, $h = (b - a) / 5$

(b) Using the trapezium rule obtained in (a) (ii), approximate $\int$(1 to 7) $x^3 / (1 + x^4)$ dx, correct to three decimal places.

$a = 1$, $b = 7$, 6 ordinates, $h = (7 - 1) / 5 = 1.2$.

x: 1, 2.2, 3.4, 4.6, 5.8, 7.

$y = x^3 / (1 + x^4)$: 0.5, 0.241, 0.116, 0.063, 0.038, 0.025.

Trapezium: $(1.2/2) \times (0.5 + 2(0.241 + 0.116 + 0.063 + 0.038) + 0.025) \approx 0.6 \times (0.5 + 0.916 + 0.025) \approx 0.6 \times 1.441 \approx 0.865$.

Answer (b): 0.865

(c) Evaluate the actual integral $\int$(1 to 7) $x^3 / (1 + x^4)$ dx and then calculate the relative error in the approximation obtained in (b). (Give your answers correct to three decimal places)

$\int x^3 / (1 + x^4)\, dx$: $u = 1 + x^4$, $du = 4x^3\, dx$, $dx = du / (4x^3)$.

$\int x^3 / (1 + x^4)\, dx = (1/4) \int 1/u\, du = (1/4) \ln|1 + x^4|$.

From 1 to 7: $(1/4) [\ln(1 + 7^4) - \ln(1 + 1^4)] = (1/4) [\ln 2402 - \ln 2] \approx (1/4) \times 7.089 \approx 1.772$.

Relative error $= |1.772 - 0.865| / 1.772 \approx 0.512$.

Answer (c): Actual = 1.772, Relative error = 0.512

8. (a) The vertices of rectangle ABCD are given by points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$. Derive the formula to calculate the area of the rectangle.

Rectangle: $AB \perp AD$, $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, $AD = \sqrt{(x_4 - x_1)^2 + (y_4 - y_1)^2}$.

Area = AB x AD.

Answer (a): Area = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \times \sqrt{(x_4 - x_1)^2 + (y_4 - y_1)^2}$

(b) Use the formula obtained in (a) to find the area of the rectangle whose vertices are the points A(1,1), B(3,5), C(-2,4) and D(-1,-5).

$AB = \sqrt{(3 - 1)^2 + (5 - 1)^2} = \sqrt{4 + 16} = \sqrt{20}$.

$AD = \sqrt{(-1 - 1)^2 + (-5 - 1)^2} = \sqrt{4 + 36} = \sqrt{40}$.

Area = $\sqrt{20} \times \sqrt{40} = \sqrt{800} = 20\sqrt{2} \approx 28.284$.

Answer (b): 28.284

(c) Show that the line $3x - 4y + 14 = 0$ is a tangent to a circle $x^2 + y^2 + 4x + 6y - 3 = 0$.

Circle: $x^2 + y^2 + 4x + 6y - 3 = 0 \rightarrow (x + 2)^2 + (y + 3)^2 = 16$, center (-2, -3), radius 4.

Line: $3x - 4y + 14 = 0 \rightarrow y = (3/4)x + 7/2$.

Distance from center to line: $|3(-2) - 4(-3) + 14| / \sqrt{3^2 + 4^2} = |-6 + 12 + 14| / 5 = 20 / 5 = 4$.

Distance = radius → tangent.

Answer (c): Shown (distance = radius)

9. (a) If $I_n = \int \sec^n x \, dx$, obtain a reduction formula for $I_n$ in terms of $I_{n-2}$ and use it to integrate $\int \sec^3 x \, dx$.

$I_n = \int \sec^n x \, dx$.

$\int \sec^n x \, dx = \sec^{n-1} x \tan x - \int \tan x \, (n-1) \sec^{n-2} x \tan x \, dx$.

Use $\tan^2 x = \sec^2 x - 1$: $I_n = \sec^{n-1} x \tan x - (n-1) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$.

$I_n = \sec^{n-1} x \tan x - (n-1) I_n + (n-1) I_{n-2}$.

$(n I_n) = \sec^{n-1} x \tan x + (n-1) I_{n-2} \rightarrow I_n = [\sec^{n-1} x \tan x + (n-1) I_{n-2}] / n$.

For $n = 3$: $I_3 = [\sec^2 x \tan x + 2 I_1] / 3$, $I_1 = \int \sec x\, dx = \ln|\sec x + \tan x|$.

$I_3 = (\sec^2 x \tan x) / 3 + (2/3) \ln|\sec x + \tan x| + C$.

Answer (a): $I_n = [\sec^{(n-1)} x \tan x + (n-1) I_{(n-2)}] / n$,

$\int \sec^3 x\, dx = (\sec^2 x \tan x) / 3 + (2/3) \ln|\sec x + \tan x| + C$

(b) Find the length of the arc given by $x = a(\cos\theta + \theta \sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$ between $\theta = 0$ and $\theta = 2\pi$.

$dx/d\theta = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$, $dy/d\theta = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta$.

Arc length: $\int \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta = \int(0 \text{ to } 2\pi) \sqrt{(a\theta\cos\theta)^2 + (a\theta\sin\theta)^2}\, d\theta = a \int(0 \text{ to } 2\pi)\, \theta\, d\theta$.

$= a [\theta^2/2] (0 \text{ to } 2\pi) = a (4\pi^2/2) = 2a\pi^2$.

Answer (b): $2a\pi^2$

10. (a) If $y = (1 - x^2) / (1 + x^2)$, show that $(1 - x^4) dy/dx + 4xy = 0$.

$y = (1 - x^2) / (1 + x^2)$.

$dy/dx = [(1 + x^2)(-2x) - (1 - x^2)(2x)] / (1 + x^2)^2 = (-2x - 2x^3 - 2x + 2x^3) / (1 + x^2)^2 = -4x / (1 + x^2)^2$.

$(1 - x^4) dy/dx = (1 - x^4)(-4x) / (1 + x^2)^2$.

$4xy = 4x (1 - x^2) / (1 + x^2)$.

Left: $(1 - x^4)(-4x) / (1 + x^2)^2 = -4x (1 - x^4) / (1 + x^2)^2$.

Right: $4x (1 - x^2) / (1 + x^2)$.

Adjust: $(1 - x^4) = (1 - x^2)(1 + x^2)$, so left $= -4x (1 - x^2) / (1 + x^2) = -$right → proven.

10. (b) If the minimum value of $f(x) = 2x^3 + 3x^2 - 12x + k$ is one-tenth of its maximum value, find the value of k.

$f(x) = 2x^3 + 3x^2 - 12x + k$.

$f'(x) = 6x^2 + 6x - 12 = 0 \rightarrow x^2 + x - 2 = 0 \rightarrow x = 1, x = -2$.

$f''(x) = 12x + 6$: at $x = 1$, $f'' = 18 > 0$ (minimum); at $x = -2$, $f'' = -18 < 0$ (maximum).

Minimum: $f(1) = 2 + 3 - 12 + k = -7 + k$.

Maximum: $f(-2) = -16 - 12 + 24 + k = -4 + k$.

$(-7 + k) = (1/10)(-4 + k)$.

$-70 + 10k = -4 + k \rightarrow 9k = 66 \rightarrow k = 22/3$.

Answer (b): $k = 22/3$

10. (c) (i) If $f(x,y) = x^2 + y + e^{(xy)}$, find $\partial f/\partial x$ and $\partial f/\partial y$.

$\partial f/\partial x = 2x + ye^{(xy)}$.

$\partial f/\partial y = 1 + xe^{(xy)}$.

Answer (i): $\partial f/\partial x = 2x + ye^{(xy)}$, $\partial f/\partial y = 1 + xe^{(xy)}$

(ii) If $z = x^2 \tan^{-1}(y/x)$, find $\partial^2 z/\partial x\partial y$ at (1,1).

$z = x^2 \tan^{-1}(y/x)$.

$\partial z/\partial y = x^2 (1 / (1 + (y/x)^2)) (1/x) = x / (1 + (y/x)^2)$.

$\partial/\partial x [\partial z/\partial y] = \partial/\partial x [x / (1 + (y/x)^2)] = [(1 + (y/x)^2) - x(-2(y/x)(y/x^2))] / (1 + (y/x)^2)^2$.

At (1,1): $(1 + 1) - 1(-2(1)(1)) / (1 + 1)^2 = (2 + 2) / 4 = 1$.

Answer (ii): 1