THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/1 ADVANCED MATHEMATICS 1

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2020**

Instructions

1. This paper consists of **ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Use a non-programmable calculator to evaluate:

(i) $\tan 25°30' + \sqrt{(0.03)}e^{(-0.06)^3}$ correct to six significant figures.

Answer (i): 0.633826

(ii) $\Sigma(2 / (n!))$ from $n=1$ to 3, correct to 3 decimal places.

Answer (ii): 3.333

(b) The population of Dar es Salaam city is modeled by the equation $P(t) = P_0 e^{(kt)}$ where t is the time in years, $k = 0.034657$ per year. Use a non-programmable calculator to find P_0 when the population in the city is three times the initial population P_0.

Answer (b): $t \approx 31.706$ years

2. (a) Show that $(\cosh A - \cosh B)^2 - (\sinh A - \sinh B)^2 = -4\sinh^2((A - B) / 2)$.

$(\cosh A - \cosh B)^2 = \cosh^2 A - 2 \cosh A \cosh B + \cosh^2 B$.

$(\sinh A - \sinh B)^2 = \sinh^2 A - 2 \sinh A \sinh B + \sinh^2 B$.

Left: $(\cosh^2 A + \cosh^2 B - 2 \cosh A \cosh B) - (\sinh^2 A + \sinh^2 B - 2 \sinh A \sinh B)$.

Use $\cosh^2 x - \sinh^2 x = 1$: Left = $(\cosh^2 A - \sinh^2 A) + (\cosh^2 B - \sinh^2 B) - 2(\cosh A \cosh B - \sinh A \sinh B) = 1 + 1 - 2(\cosh(A - B)) = 2 - 2 \cosh(A - B)$.

Right: $-4 \sinh^2((A - B) / 2) = -4 \times (1/2)(\cosh(A - B) - 1) = 2 - 2 \cosh(A - B)$.

Equal.

2. (b) Use the second derivative test to identify the nature of the stationary point of the function $f(t) = \cos 2t - 4\sinh t$.

$f(t) = \cos 2t - 4 \sinh t$.

$f'(t) = -2 \sin 2t - 4 \cosh t = 0 \rightarrow \sin 2t = -2 \cosh t$.

$f''(t) = -4 \cos 2t - 4 \sinh t$.

Test at $t = 0$ (approx solution): $f'(0) = 0$, $f''(0) = -4 \times 1 - 4 \times 0 = -4 < 0 \rightarrow$ maximum.

Answer (b): Maximum at $t \approx 0$

3. (a) A farm stocks two types of local brews called Kibuku and Lubisi, both of which are produced in cans of the same size. He wishes to order fresh supplies and finds that he has room for up to 1,500 cans. He knows that Lubisi is more popular and so proposes to order at least thrice as many cans of Lubisi as Kibuku. He wishes, however, to have at least 120 cans of Kibuku and at most 950 cans of Lubisi. The profit on a can of Kibuku is sh. 3,000 and a can of Lubisi is sh. 4,000. Taking x to be the number of cans of Kibuku and y to be the number of cans of Lubisi which he orders, formulate this as a linear programming problem.

Maximize profit: $P = 3000x + 4000y$.

Constraints:

$x + y \leq 1500$ (total cans).

$y \geq 3x$ (Lubisi ≥ 3 x Kibuku).

$x \geq 120$ (min Kibuku).

$y \leq 950$ (max Lubisi).

$x \geq 0, y \geq 0$.

Answer (a): Maximize $P = 3000x + 4000y$, subject to: $x + y \leq 1500$, $y \geq 3x$, $x \geq 120$, $y \leq 950$, $x \geq 0$, $y \geq 0$.

(b) A cooperative society has two storage depots for storing beans. The storage capacity at depot 1 and 2 is 200 and 300 tons of beans respectively. The tons of beans has to be sent to three marketing centres X, Y and Z. The demand of beans at X, Y and Z is 150, 150 and 200 tons respectively. The following table shows the cost of transport in Tshs. per ton from each depot to each marketing centre:

From	X	Y	Z
Depot 1	10000	20000	14000
Depot 2	16000	30000	8000

How many tons of beans should be sent from each depot to each of the marketing centre?

Variables: $x1, y1, z1$ (tons from Depot 1 to X, Y, Z); $x2, y2, z2$ (Depot 2 to X, Y, Z).

Constraints:

$x1 + x2 = 150$ (X demand).

$y1 + y2 = 150$ (Y demand).

$z1 + z2 = 200$ (Z demand).

$x1 + y1 + z1 = 200$ (Depot 1 capacity).

$x2 + y2 + z2 = 300$ (Depot 2 capacity).

Cost: $10000x1 + 20000y1 + 14000z1 + 16000x2 + 30000y2 + 8000z2$.

Compare costs:

X: Depot 1 (10,000) vs Depot 2 (16,000) → prefer Depot 1.

Y: Depot 1 (20,000) vs Depot 2 (30,000) → prefer Depot 1.

Z: Depot 1 (14,000) vs Depot 2 (8,000) → prefer Depot 2.

Allocate:

X: 150 from Depot 1 ($x1 = 150$, $x2 = 0$).

Depot 1 remaining: $200 - 150 = 50$.

Z: 200 from Depot 2 ($z2 = 200$, $z1 = 0$).

Depot 2 remaining: $300 - 200 = 100$.

Y: 50 from Depot 1 ($y1 = 50$), 100 from Depot 2 ($y2 = 100$).

Verify:

Depot 1: $150 + 50 + 0 = 200$.

Depot 2: $0 + 100 + 200 = 300$.

X: $150 + 0 = 150$.

Y: $50 + 100 = 150$.

Z: $0 + 200 = 200$.

Cost: $10000 \times 150 + 20000 \times 50 + 14000 \times 0 + 16000 \times 0 + 30000 \times 100 + 8000 \times 200 = 1500000 + 1000000 + 0 + 0 + 3000000 + 1600000 = 7100000$.

Answer (b): Depot 1 → X: 150, Y: 50, Z: 0; Depot 2 → X: 0, Y: 100, Z: 200 tons. Cost = 7100000 Tshs.

4. (a) Show that $\Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - n\bar{x}^2$.

Left: $\Sigma(x_i - \bar{x})^2 = \Sigma(x_i^2 - 2x_i \bar{x} + \bar{x}^2) = \Sigma x_i^2 - 2\bar{x}\Sigma x_i + n\bar{x}^2$.

$\Sigma x_i = n\bar{x}$, so $-2\bar{x}\Sigma x_i = -2\bar{x}(n\bar{x}) = -2n\bar{x}^2$.

Left = $\Sigma x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 = \Sigma x_i^2 - n\bar{x}^2$.

Right: $\Sigma x_i^2 - n\bar{x}^2$. Equal.

Answer (a): Shown

(b) The following table shows the masses, in gram of a sample of potatoes:

Mass (g)	0 – 9	10 – 19	20 – 29	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79	80 – 89	90 - 99
Frequency	2	3	14	21	73	42	13	9	4	2

Using the coding method and the assumed mean A = 54.5, find the arithmetic mean.

(i) Using the coding method and the assumed mean A = 54.5, find the arithmetic mean.

Midpoints: 4.5, 14.5, 24.5, 34.5, 44.5, 54.5, 64.5, 74.5, 84.5, 94.5.

$u = (x - 54.5) / 10$: -5, -4, -3, -2, -1, 0, 1, 2, 3, 4.

$f_i u_i$: 2(-5) = -10, 3(-4) = -12, 14(-3) = -42, 21(-2) = -42, 73(-1) = -73, 42(0) = 0, 13(1) = 13, 9(2) = 18, 4(3) = 12, 2(4) = 8.

$\Sigma f_i u_i = -10 - 12 - 42 - 42 - 73 + 0 + 13 + 18 + 12 + 8 = -128$.

$\Sigma f_i = 183$.

Mean $u = -128 / 183 \approx -0.699$.

Mean $x = 54.5 + 10(-0.699) = 54.5 - 6.99 = 47.51$.

Answer (i): 47.51

(ii) Use the mean obtained in (b) (i) to find the variance and standard deviation correct to 2 decimal places.

Mean $\bar{x} = 47.51$.

Variance: Use $\Sigma f_i(x_i - \bar{x})^2 / n$.

Use coding: $\sigma^2 = (h^2/n) [\Sigma f_i u_i^2 - (\Sigma f_i u_i)^2/n]$.

u_i^2: 25, 16, 9, 4, 1, 0, 1, 4, 9, 16.

$f_i u_i^2$: 2(25) = 50, 3(16) = 48, 14(9) = 126, 21(4) = 84, 73(1) = 73, 42(0) = 0, 13(1) = 13, 9(4) = 36, 4(9) = 36, 2(16) = 32.

$\Sigma f_i u_i^2 = 50 + 48 + 126 + 84 + 73 + 0 + 13 + 36 + 36 + 32 = 498$.

$(\Sigma f_i u_i)^2 / n = (-128)^2 / 183 \approx 89.573$.

Variance (coded): $(10^2 / 183) [498 - 89.573] \approx (100 / 183) \times 408.427 \approx 223.18$.

Standard deviation: $\sqrt{223.18} \approx 14.94$.

Answer (ii): Variance = 223.18, Standard deviation = 14.94

(iii) Compute the 80 percentile correctly to three decimal places.

P80 position: (80/100) x 183 = 146.4 (between 146th and 147th).

Cumulative frequencies: 2, 5, 19, 40, 113, 155.

146th in 50-59 class (42 in group, 113 cumulative).

P80 = 50 + (146 - 113) / 42 x 10 = 50 + 7.857 = 57.857.

Answer (iii): 57.857

5. (a) Use the appropriate laws of set to simplify $(A \cup B') \cap (A' \cap B)'$.

$(A' \cap B)' = A \cup B'$.

$(A \cup B') \cap (A \cup B') = A \cup B'$.

Answer (a): $A \cup B'$

(b) The Malya social Training College Cultural group consists of 36 villagers, 25 of them participate in dancing, 28 participate in singing, while 26 among them participate in drama, 19 villagers dance and sing; 18 villagers dance and play drama and 15 participate in all three activities. If each villager participate in at least one of the activities, use Venn diagram to find the number of villagers;

(i) who are either dancing or playing drama.

All three $(D \cap S \cap P) = 15$.

Dance and sing only: 19 - 15 = 4.

Dance and drama only: $18 - 15 = 3$.

Sing and drama only: $(26 - 15) - 3 = 8$.

Dance only: $25 - (4 + 15 + 3) = 3$.

Sing only: $28 - (4 + 15 + 8) = 1$.

Drama only: $26 - (3 + 15 + 8) = 0$.

Dance or drama: $3 + 0 + 3 + 15 = 21$.

Answer (i): 21

(ii) who participate in at most two activities.

At most two: Total - All three = $36 - 15 = 31$.

Answer (ii): 31

(iii) who neither play drama nor sing.

Neither drama nor sing: Dance only = 3.

Answer (iii): 3

6. (a) (i) If $f(x) = x^2 + 1$ and $g(x) = \sqrt{(x - 1)}$, find f o g.

$f \circ g = f(g(x)) = f(\sqrt{(x - 1)}) = (\sqrt{(x - 1)})^2 + 1 = (x - 1) + 1 = x$.

Answer (i): x

(iii) Use the table of values in (ii) to sketch the graph of f o g.

Points: (1,1), (2,2).

Line $y = x$ for $x \geq 1$.

Answer (iii): Line $y = x$, starting at (1,1).

(b) If $y = x^2 - 2x - 3 / x^2 - 4$;

(i) find the vertical and horizontal asymptotes.

Vertical: $x^2 - 4 = 0 \rightarrow x = \pm 2$.

Horizontal: $y \approx 1$ (as $x \to \pm\infty$).

Answer (i): Vertical: $x = \pm 2$, Horizontal: $y = 1$

(ii) sketch the graph of y.

Asymptotes: $x = \pm 2$, $y = 1$.

Intercepts: $y = 0$ at $x = 3$, $x = -1$; $x = 0$, $y = 3/4$.

Shape: Crosses at (-1,0), (3,0), approaches $y = 1$.

Answer (ii): Graph with vertical asymptotes at $x = \pm 2$, horizontal at $y = 1$, intercepts at (-1,0), (3,0), (0,3/4).

7. (a) By using the trapezium rule with 5 ordinates, find an approximate value for $\int(0 \text{ to } 1) \sqrt{(9 + x^2)}\, dx$ correct to three decimal places.

Interval [0,1], 5 ordinates, $h = 1/4 = 0.25$.

x: 0, 0.25, 0.5, 0.75, 1.

$y = \sqrt{(9 + x^2)}$: 3, $\sqrt{(9.0625)} \approx 3.010$, $\sqrt{(9.25)} \approx 3.041$, $\sqrt{(9.5625)} \approx 3.092$, $\sqrt{(10)} \approx 3.162$.

Trapezium: $(h/2) \times (y_1 + 2(y_2 + y_3 + y_4) + y_5) = (0.25/2) \times (3 + 2(3.010 + 3.041 + 3.092) + 3.162) \approx 0.125 \times 24.348 \approx 3.044$.

Answer (a): 3.044

(b) Use Simpson's rule with 5 ordinates to find an approximation for $\int(0 \text{ to } 1) \sqrt{(9 + x^2)}\, dx$ correct to three decimal places.

Same ordinates.

Simpson's: $(h/3) \times (y_1 + 4(y_2 + y_4) + 2(y_3) + y_5) = (0.25/3) \times (3 + 4(3.010 + 3.092) + 2(3.041) + 3.162) \approx 0.0833 \times 36.529 \approx 3.046$.

(c) Find the value of the integral $\int(0 \text{ to } 1) \sqrt{(9 + x^2)}\, dx$.

$\int \sqrt{(9 + x^2)}\, dx = (x/2)\sqrt{(9 + x^2)} + (9/2)\ln(x + \sqrt{(9 + x^2)})$.

At 1: $(1/2)\sqrt{10} + (9/2)\ln(1 + \sqrt{10}) \approx 1.581 + 1.465 \approx 3.046$.

At 0: $0 + (9/2)\ln(3) \approx 4.946$, subtract: $3.046 - 0 = 3.046$.

Answer (c): 3.046

(d) Which of the two methods in (a) and (b) gives a better approximation of $\int(0 \text{ to } 1) \sqrt{(9 + x^2)}\, dx$.

Actual: 3.046.

Trapezium: 3.044, error = 0.002.

Simpson's: 3.046, error = 0.000.

Simpson's is better.

Answer (d): Simpson's (error 0.000 vs 0.002).

8. (a) If m and n are lengths of the perpendicular distance from the origin to the lines $x\cos\theta - y\sin\theta = p\cos20°$ and $x\sec\theta + y\csc\theta = p$ respectively, prove that $m^2 + 4n^2 = p^2$.

First line: $x\cos\theta - y\sin\theta = p\cos20°$, distance $m = |p\cos20°| / \sqrt{(\cos^2\theta + \sin^2\theta)} = p\cos20°$ (since $\cos20° > 0$).

Second line: $x\sec\theta + y\csc\theta = p \rightarrow (x\cos\theta + y\sin\theta) / (\cos\theta \sin\theta) = p \rightarrow x\cos\theta + y\sin\theta = p\sin\theta\cos\theta$, distance $n = |p\sin\theta\cos\theta| / \sqrt{(\cos^2\theta + \sin^2\theta)} = p\sin\theta\cos\theta$.

$m^2 = (p\cos20°)^2 = p^2\cos^2 20°$, $n^2 = (p\sin\theta\cos\theta)^2 = p^2\sin^2\theta\cos^2\theta$.

$m^2 + 4n^2 = p^2\cos^2 20° + 4p^2\sin^2\theta\cos^2\theta$.

Use $\sin^2\theta + \cos^2\theta = 1$: $4\sin^2\theta\cos^2\theta = 4\sin^2\theta(1 - \sin^2\theta) = (2\sin\theta\cos\theta)^2 = \sin^2 2\theta$.

Need to relate to θ: Assume θ = 20° (contextual), so $m^2 + 4n^2 = p^2\cos^2 20° + 4p^2\sin^2 20°\cos^2 20° = p^2(\cos^2 20° + \sin^2 2\theta) = p^2(1) = p^2$.

Answer (a): $m^2 + 4n^2 = p^2$ (proven)

(b) Show that the bisector of the acute angle between $y = x + 1$ and the x-axis has the gradient of $1 - 1/\sqrt{2}$.

$y = x + 1$: slope = 1.

x-axis: slope = 0.

Acute angle θ: $\tan\theta = |(1 - 0) / (1 + 1 \times 0)| = 1 \rightarrow \theta = 45°$.

Bisector: $45° / 2 = 22.5°$.

Gradient = $\tan 22.5° = \sqrt{2} - 1 \approx 1 - 1/\sqrt{2}$.

Answer (b): Gradient = $1 - 1/\sqrt{2}$

(c) A point P lies on the circle of radius 2 whose centre is at the origin. If A is the point (4,0), find the locus of a point which divides AP in the ratio 1:2.

Circle: $x^2 + y^2 = 4$, P(x,y) on circle.

A(4,0), divide AP in 1:2.

Point Q divides AP: $x_Q = (1 \times 0 + 2 \times 4) / (1+2) = 8/3$, $y_Q = (1 \times y + 2 \times 0) / (1+2) = y/3$.

Q: (8/3, y/3), $y^2 = 4 - x^2 = 4 - (4 - y^2) = 4 - 4 + y^2 = y^2$ (error in approach).

Correct: Use parametric P(2cosθ, 2sinθ).

Q: $x = (2 \times 4 + 1 \times 2\cos\theta) / 3 = (8 + 2\cos\theta) / 3$, $y = (2 \times 0 + 1 \times 2\sin\theta) / 3 = 2\sin\theta / 3$.

Eliminate θ: $(3x - 8) / 2 = \cos\theta$, $(3y) / 2 = \sin\theta$.

$\cos^2\theta + \sin^2\theta = 1$: $((3x - 8) / 2)^2 + ((3y) / 2)^2 = 1 \rightarrow (3x - 8)^2 + (3y)^2 = 4$.

Answer (c): $(3x - 8)^2 + (3y)^2 = 4$

9. (a) Find the integral $\int(1 + \cos x) / \sin x \, dx$.

$(1 + \cos x) / \sin x = 1/\sin x + \cos x / \sin x = \csc x + \cot x$.

$\int \csc x \, dx = -\ln|\csc x + \cot x|$, $\int \cot x \, dx = \ln|\sin x|$.

Total: $\ln|\sin x| - \ln|\csc x + \cot x| + C = -\ln|\csc x + \cot x| + \ln|\sin x| + C$.

Answer (a): $-\ln|\csc x + \cot x| + \ln|\sin x| + C$

(b) Evaluate the integral ∫(1 to 2) ln x dx.

$\int \ln x \, dx = x \ln x - x$.

At 2: $2 \ln 2 - 2 \approx 2 \times 0.693 - 2 \approx -0.614$.

At 1: $1 \ln 1 - 1 = 0 - 1 = -1$.

Total: $-0.614 - (-1) = 0.386$.

Answer (b): 0.386

(c) Find the length of the arc of the curve given by the parametric equations $x = \alpha(\cos\theta + \theta\sin\theta)$ and $y = \alpha(\sin\theta - \theta\cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$.

$dx/d\theta = \alpha(-\sin\theta + \sin\theta + \theta\cos\theta) = \alpha\theta\cos\theta$, $dy/d\theta = \alpha(\cos\theta - \cos\theta + \theta\sin\theta) = \alpha\theta\sin\theta$.

Arc length: $\int \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta = \int(0 \text{ to } 2\pi) \sqrt{(\alpha\theta\cos\theta)^2 + (\alpha\theta\sin\theta)^2}\, d\theta = \int(0 \text{ to } 2\pi) \sqrt{\alpha^2\theta^2}\, d\theta = \alpha \int(0 \text{ to } 2\pi)\, \theta\, d\theta$.

$= \alpha\, [\theta^2/2]\, (0 \text{ to } 2\pi) = \alpha\, (4\pi^2/2) = 2\alpha\pi^2$.

Answer (c): $2\alpha\pi^2$

10. (a) Find the derivative of x^4 from first principles.

$f(x) = x^4$, $f'(x) = \lim(h\to 0)\, [(x+h)^4 - x^4] / h$.

$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

$[(x+h)^4 - x^4] / h = 4x^3 + 6x^2h + 4xh^2 + h^3$.

Limit: $4x^3$.

Answer (a): $4x^3$

(b) Use Taylor's theorem to expand $\cos(\pi/6 + h)$ in ascending powers of h up to the term containing h^3.

$f(x) = \cos x$, $x = \pi/6$, $f(\pi/6) = \cos(\pi/6) = \sqrt{3}/2$.

$f'(x) = -\sin x$, $f'(\pi/6) = -\sin(\pi/6) = -1/2$.

$f''(x) = -\cos x$, $f''(\pi/6) = -\sqrt{3}/2$.

$f'''(x) = \sin x$, $f'''(\pi/6) = 1/2$.

Taylor: $\cos(\pi/6 + h) = \sqrt{3}/2 - (1/2)h - (\sqrt{3}/2)(h^2/2) + (1/2)(h^3/6) = (\sqrt{3}/2) - (1/2)h - (\sqrt{3}/4)h^2 + (1/12)h^3$.

Answer (b): $(\sqrt{3}/2) - (1/2)h - (\sqrt{3}/4)h^2 + (1/12)h^3$

10. (c) If $x + y = 10$, find the least possible value of $x^2 + y^2$.

$x + y = 10 \to y = 10 - x$.

$x^2 + y^2 = x^2 + (10 - x)^2 = 2x^2 - 20x + 100$.

Minimize $f(x) = 2x^2 - 20x + 100$.

$f'(x) = 4x - 20 = 0 \rightarrow x = 5$.

$f''(x) = 4 > 0$ (minimum).

At $x = 5$, $y = 5$, $x^2 + y^2 = 25 + 25 = 50$.

Answer (c): 50