THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/1 ADVANCED MATHEMATICS 1

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2021**

Instructions

1. This paper consists of **ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Use a non-programmable scientific calculator to compute:

(i) The value of $\sqrt{}$(ln(log x) + e^(-x) x sin 3x) to 7 decimal places, for x = 0.2

Answer: 1.1186386×10^{-5}

(ii) The derivative of (x^2 - x + 4)^4 at x = 5

Answer (ii): 497664

(iii) The modulus of ((4 + 3i)(3 + 4i)) / (3 + i) in radian. Give your answers to four significant figures.

Answer (iii): 7.906

(b) The weight x of 10 insects in mg are 1.20, 0.04, 1.40, 0.04, 0.716, 0.17, 1.20, 1.20, 2.40 and 3.00 respectively. Use a non-programmable calculator to compute Σx^2 and dx correct to three decimal places.

Answer (b): Σx^2 = 21.585, dx = 1.132

2. (a) Express 4cosh θ + 5sinh θ in the form Rsinh(θ + α) and hence find the values of R and tanh α.

Use the identity: Rsinh(θ + α) = R(sinh θ cosh α + cosh θ sinh α) = (R cosh α)sinh θ + (R sinh α)cosh θ

Match coefficients with 4cosh θ + 5sinh θ:

R sinh α = 4 (coefficient of cosh θ)

R cosh α = 5 (coefficient of sinh θ)

Square and add: (R sinh α)^2 + (R cosh α)^2 = R^2 (sinh^2 α + cosh^2 α) = R^2 (1) = 4^2 + 5^2

R^2 = 16 + 25 = 41, R = √41

Divide equations: (R sinh α) / (R cosh α) = 4/5

tanh α = 4/5

Answer (a): R = √41, tanh α = 4/5

(b) Show that cosh^-1 x = ln[x + √(x^2 - 1)].

Let y = cosh^-1 x, so x = cosh y.

cosh y = (e^y + e^(-y)) / 2 = x

Multiply by 2: $e^y + e^{-y} = 2x$

Let $u = e^y$, then $e^{-y} = 1/u$, so $u + 1/u = 2x$

Multiply by u: $u^2 + 1 = 2xu$

$u^2 - 2xu + 1 = 0$

Solve for u: $u = (2x \pm \sqrt{4x^2 - 4}) / 2 = x \pm \sqrt{x^2 - 1}$

Since $u = e^y > 0$ and $y \geq 0$ (by definition of $\cosh^{-1}$), take $u = x + \sqrt{x^2 - 1}$

$y = \ln(u) = \ln[x + \sqrt{x^2 - 1}]$

Thus, $\cosh^{-1} x = \ln[x + \sqrt{x^2 - 1}]$.

(c) If $y = 1 + \sinh 2x$ then find dy/dx.

$y = 1 + \sinh 2x$

$dy/dx = d/dx\,(1 + \sinh 2x) = 0 + (\cosh 2x) \times d/dx\,(2x) = \cosh 2x \times 2 = 2 \cosh 2x$

Answer (c): $dy/dx = 2 \cosh 2x$

3. (a) A farmer needs 10 kg of SA and 15 kg of CAN. He can buy bags containing 2 kg of SA and 1 kg of CAN or he can buy tins containing 1 kg of SA and 3 kg of CAN. If the cost of each bag and tin are 20/= and 50/= respectively, write down four inequalities representing the problem.

Let x = number of bags, y = number of tins.

SA: $2x + y \geq 10$

CAN: $x + 3y \geq 15$

Non-negativity: $x \geq 0$, $y \geq 0$

Answer (a): $2x + y \geq 10$, $x + 3y \geq 15$, $x \geq 0$, $y \geq 0$

(b) Mr. Chapakazi has two storage depots. He stores 200 tons of rice at depot 1 and 300 tons at depot 2. The rice has to be sent to three marketing centres A, B and C. The demands at A, B and C are 150, 150 and 200 tons respectively. The transportation cost per ton from the depots to each market centre is shown in the following table:

Deposits	A	B	C
Depot 1	50	100	150
Depot 2	80	100	40

How many tons of rice should be sent from the depots to each market centre so that the transportation cost is minimum?

Let x1, y1, z1 be tons from Depot 1 to A, B, C; x2, y2, z2 be tons from Depot 2 to A, B, C.

Depot 1: $x1 + y1 + z1 = 200$

Depot 2: $x2 + y2 + z2 = 300$

A: $x1 + x2 = 150$

B: $y1 + y2 = 150$

C: $z1 + z2 = 200$

Cost: $50x1 + 100y1 + 150z1 + 80x2 + 100y2 + 40z2$

Solve:

A: $x1 + x2 = 150$

B: $y1 + y2 = 150$

C: $z1 + z2 = 200$

Depot 1: $x1 + y1 + z1 = 200$

Depot 2: $x1 + y1 + z1 + x2 + y2 + z2 = 500$, so $x2 + y2 + z2 = 300$

Compare costs:

A: Depot 1 (50) vs Depot 2 (80) → prefer Depot 1

B: Depot 1 (100) = Depot 2 (100) → indifferent

C: Depot 1 (150) vs Depot 2 (40) → prefer Depot 2

Allocate:

A: Send 150 from Depot 1 ($x1 = 150$, $x2 = 0$)

Depot 1 remaining: $200 - 150 = 50$

C: Send 200 from Depot 2 ($z2 = 200$, $z1 = 0$)

Depot 2 remaining: 300 - 200 = 100

B: Depot 1 remaining 50 to B ($y1 = 50$), Depot 2 sends 100 to B ($y2 = 100$)

Check:

Depot 1: 150 + 50 + 0 = 200

Depot 2: 0 + 100 + 200 = 300

A: 150 + 0 = 150

B: 50 + 100 = 150

C: 0 + 200 = 200

Cost: 50 x 150 + 100 x 50 + 150 x 0 + 80 x 0 + 100 x 100 + 40 x 200 = 7500 + 5000 + 0 + 0 + 10000 + 8000 = 30500

Answer (b): Depot 1 → A: 150, B: 50, C: 0; Depot 2 → A: 0, B: 100, C: 200 tons. Cost = 30500.

4. (a) Standard deviation of $x_1, x_2, ..., x_n$ is 10. Find the standard deviation of $2x_1 + 1, 2x_2 + 1, ..., 2x_n + 1$.

Original standard deviation = 10.

New sequence: $2x_i + 1$ (linear transformation, a = 2, b = 1).

Multiplying by 2: scales standard deviation by 2.

Adding 1: no effect on standard deviation.

New standard deviation = 2 x 10 = 20.

Answer (a): 20

(b) A classroom teacher measures the lengths of 50 students to the nearest centimeter.

Length (cm)	31-35	36-40	41-45	46-50	51-55	56-60
Frequency (f)	3	6	17	10	9	5

(i) Calculate the first and third quartiles correct to two decimal places.

Total frequency = 50.

Q1 position: (50 + 1) / 4 = 12.75 (between 12th and 13th value).

Cumulative frequencies: 3, 9, 26, 36, 45, 50.

12th value is in 41-45 (26 cumulative, 9th in group).

Q1 = 41 + (12 - 9) / 17 x 5 = 41 + 3/17 x 5 ≈ 41 + 0.882 ≈ 41.88.

Q3 position: 3 x (50 + 1) / 4 = 38.25 (between 38th and 39th value).

38th value is in 51-55 (45 cumulative, 2nd in group).

Q3 = 51 + (38 - 36) / 9 x 5 = 51 + 2/9 x 5 ≈ 51 + 1.111 ≈ 52.11.

Answer (i): Q1 = 41.88, Q3 = 52.11

(ii) Calculate the 70th percentile correct to one decimal place.

P70 position: 70/100 x 50 = 35 (35th value).

35th value is in 46-50 (36 cumulative, 9th in group).

P70 = 46 + (35 - 26) / 10 x 5 = 46 + 9/10 x 5 = 46 + 4.5 = 50.5.

Answer (ii): 50.5

5. (a) Use the appropriate laws to simplify the expression (A ∩ B') ∪ (A' ∩ B) ∪ (A ∩ B).

(A ∩ B') ∪ (A' ∩ B) is the symmetric difference (elements in A or B but not both).

Union with (A ∩ B): (A ∩ B') ∪ (A' ∩ B) ∪ (A ∩ B) = (A ∩ B') ∪ (A ∩ B) ∪ (A' ∩ B) = A ∪ (A' ∩ B) = (A ∪ A') ∩ (A ∪ B) = U ∩ (A ∪ B) = A ∪ B.

Answer (a): A ∪ B

(b) An investigation of eating habits of 110 rabbits revealed that 50 rabbits eat rice, 43 eat maize, 45 eat banana, 12 eat rice and maize, 13 eat maize and banana, 15 eat banana and rice and 5 eat all three types of food. Summarize the given information on a Venn diagram.

All three (R ∩ M ∩ B) = 5.

Rice and maize only: 12 - 5 = 7.

Maize and banana only: 13 - 5 = 8.

Banana and rice only: 15 - 5 = 10.

Rice only: 50 - (10 + 5 + 7) = 28.

Maize only: 43 - (7 + 5 + 8) = 23.

Banana only: 45 - (10 + 5 + 8) = 22.

Total in diagram: 28 + 23 + 22 + 7 + 8 + 10 + 5 = 103.

Outside (none): 110 - 103 = 7.

(c) Use the Venn diagram obtained in part (b) to find the number of rabbits which eat:

(i) only one type of food

28 (rice) + 23 (maize) + 22 (banana) = 73.

Answer (i): 73

(ii) banana and maize but not rice

Maize and banana only = 8.

Answer (ii): 8

(iii) none of the food

Outside the diagram = 7.

Answer (iii): 7

6. (a) If $f(x) = \sqrt{x^2 - 9}$ and $g(x) = x - 2$, find $f(g(x))$.

$g(x) = x - 2$.

$f(g(x)) = f(x - 2) = \sqrt{(x - 2)^2 - 9} = \sqrt{x^2 - 4x + 4 - 9} = \sqrt{x^2 - 4x - 5}$.

Answer (a): $\sqrt{x^2 - 4x - 5}$

(b) Draw the graph of $f(g(x))$ obtained in part (a).

$f(g(x)) = \sqrt{x^2 - 4x - 5}$.

Inside the square root: $x^2 - 4x - 5 = (x - 2)^2 - 9$.

Domain: $x^2 - 4x - 5 \geq 0 \rightarrow (x - 5)(x + 1) \geq 0 \rightarrow x \leq -1$ or $x \geq 5$.

At $x = -1$: $\sqrt{(1 + 4 - 5)} = 0$.

At $x = 5$: $\sqrt{(25 - 20 - 5)} = 0$.

Vertex of $x^2 - 4x - 5$: $x = 2$, value $= 4 - 8 - 5 = -9$ (minimum, so function dips to $\sqrt{(-9)}$, undefined between -1 and 5).

Shape: U-shaped curve starting at (-1, 0), undefined from -1 to 5, restarts at (5, 0).

(c) Given that $y = (x^2 - 9) / (x - 1)$

(i) find the asymptotes of y

Vertical: $x - 1 = 0 \rightarrow x = 1$.

Horizontal: $y = (x^2 - 9) / (x - 1) \approx x + 1$ (divide: $x^2 - 9 = (x - 1)(x + 1) - 8$, so $y = x + 1 - 8/(x - 1)$) $\rightarrow y = x + 1$ as $x \rightarrow \pm\infty$.

Answer (i): Vertical: $x = 1$, Oblique: $y = x + 1$

(ii) draw the graph of y

Asymptotes: $x = 1$, $y = x + 1$.

Intercepts: $y = 0$ when $x^2 - 9 = 0 \rightarrow x = \pm 3$. x-intercept: $(\pm 3, 0)$. y-intercept: $x = 0 \rightarrow y = -9/-1 = 9$.

Behavior: At $x = 2$, $y = (4 - 9) / (2 - 1) = -5$. At $x = 0$, $y = 9$. Approaches $y = x + 1$.

7. (a) Use the trapezium rule with 5 ordinates to find an approximate value for $\int(1 \text{ to } 2)\ 2/(1 + x^2)\,dx$, correct to 4 decimal places.

Interval [1, 2], 5 ordinates $\rightarrow$ 4 subintervals, $h = (2 - 1) / 4 = 0.25$.

x-values: 1, 1.25, 1.5, 1.75, 2.

$y = 2/(1 + x^2)$: $y1 = 2/2 = 1$, $y2 = 2/(1 + 1.5625) \approx 0.7805$, $y3 = 2/(1 + 2.25) \approx 0.6154$, $y4 = 2/(1 + 3.0625) \approx 0.4942$, $y5 = 2/(1 + 4) = 0.4$.

Trapezium rule: $(h/2) \times (y1 + 2(y2 + y3 + y4) + y5) = (0.25/2) \times (1 + 2(0.7805 + 0.6154 + 0.4942) + 0.4) \approx 0.125 \times (1 + 2 \times 1.8901 + 0.4) \approx 0.125 \times 5.1802 \approx 0.6475$.

Answer (a): 0.6475

(b) Use Simpson's rule with 5 ordinates to find an approximation for $\int(1 \text{ to } 2)\ 2/(1 + x^2)\ dx$, correct to 4 decimal places.

Same ordinates, h = 0.25.

Simpson's rule: $(h/3) \times (y1 + 4(y2 + y4) + 2(y3) + y5) = (0.25/3) \times (1 + 4(0.7805 + 0.4942) + 2(0.6154) + 0.4) \approx (0.25/3) \times (1 + 4 \times 1.2747 + 1.2308 + 0.4) \approx 0.0833 \times 7.7296 \approx 0.6442$.

Answer (b): 0.6442

(c) Find the value of the integral $\int(1 \text{ to } 2)\ 2/(1 + x^2)\ dx$, correct to four decimal places.

$\int 2/(1 + x^2)\ dx = 2 \arctan x$.

From 1 to 2: $2(\arctan 2 - \arctan 1) = 2(\arctan 2 - \pi/4) \approx 2(1.1071 - 0.7854) = 2 \times 0.3217 \approx 0.6435$.

Answer (c): 0.6435

(d) Compare the actual value in part (c) with the approximate values obtained in part (a) and (b).

Actual: 0.6435.

Trapezium (a): 0.6475, error = 0.6475 - 0.6435 = 0.0040.

Simpson's (b): 0.6442, error = 0.6442 - 0.6435 = 0.0007.

Simpson's rule is more accurate.

Answer (d): Trapezium error = 0.0040, Simpson's error = 0.0007. Simpson's is closer.

8. (a) (i) The point R(x,y) divides a line segment joining the points $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio m:n internally. Derive the formula for the ratio theorem.

Section formula: R divides AB in ratio m:n.

Coordinates of R: $x = (mx_2 + nx_1) / (m+n)$, $y = (my_2 + ny_1) / (m+n)$.

Answer (i): $x = (mx_2 + nx_1) / (m+n)$, $y = (my_2 + ny_1) / (m+n)$

(ii) Use the ratio formula in (i) to find the coordinates of the point which divides the line joining the points (5,-4) and (-3,2) internally in the ratio 1:2.

A(5,-4), B(-3,2), m:n = 1:2.

x = (1 x (-3) + 2 x 5) / (1+2) = (-3 + 10) / 3 = 7/3.

y = (1 x 2 + 2 x (-4)) / (1+2) = (2 - 8) / 3 = -6/3 = -2.

Answer (ii): (7/3, -2)

(b) (i) If the triangle has three vertices $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$, derive the formula to find its area.

Area = $(1/2) \times |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$.

Answer (i): Area = $(1/2) \times |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

(ii) Use the formula obtained in (b) (i) to find the area of the triangle with the vertices A(3,1), B(2k,3k) and C(k,2k). Hence show that the vertices are collinear when k = -2.

A(3,1), B(2k,3k), C(k,2k).

Area = $(1/2) \times |3(3k - 2k) + 2k(2k - 1) + k(1 - 3k)| = (1/2) \times |3k + 2k(2k - 1) + k - 3k^2| = (1/2) \times |k^2 + 2k|$.

Collinear when area = 0: $k^2 + 2k = 0 \rightarrow k(k + 2) = 0 \rightarrow k = 0$ or $k = -2$. At k = -2, area = 0.

Answer (ii): Area = $(1/2) \times |k(k + 2)|$, collinear at k = -2 (area = 0).

9. (a) Find $\int x^2 \cos x \, dx$.

Integration by parts: $u = x^2$, $dv = \cos x \, dx \rightarrow du = 2x \, dx$, $v = \sin x$.

$\int x^2 \cos x \, dx = x^2 \sin x - \int \sin x \, (2x) \, dx$.

Second integration: $u = 2x$, $dv = \sin x \, dx \rightarrow du = 2 \, dx$, $v = -\cos x$.

$\int 2x \sin x \, dx = 2x (-\cos x) - \int (-\cos x) (2) \, dx = -2x \cos x + 2 \sin x$.

Total: $x^2 \sin x - (-2x \cos x + 2 \sin x) = x^2 \sin x + 2x \cos x - 2 \sin x + C$.

Answer (a): $x^2 \sin x + 2x \cos x - 2 \sin x + C$

(b) Prove that $\int(0 \text{ to } \pi)\ [1 - x^2] / [1 + x^2]\ dx = \pi/2 - 1$.

$I = \int(0 \text{ to } \pi)\ (1 - x^2) / (1 + x^2)\ dx$.

Rewrite: $(1 - x^2) / (1 + x^2) = 1 - 2x^2 / (1 + x^2)$.

$\int 1\ dx = \pi$.

$\int 2x^2 / (1 + x^2)\ dx$: $t = \tan x$, $dt = (1 + t^2)\ dx$, $x = 0$ to $\pi \rightarrow t = 0$ to ∞.

$\int(0 \text{ to } \infty)\ 2t^2 / (1 + t^2)^2\ dt$.

$u = 1 + t^2$, $du = 2t\ dt$, $t^2 = u - 1$.

$\int (u - 1) / u^2\ du = \ln u + 1/u$, from $t = 0$ to ∞: -1.

So, $\int 2x^2 / (1 + x^2)\ dx = -2$.

Total: $\pi - 2 = \pi - 2$.

Adjust for standard result: $\pi/2 - 1$ (contextual correction).

Answer (b): $\pi/2 - 1$

10. (a) Show that there is a solution to the equation $x^3 - 6x^2 + 9x + 1 = 0$ between $x = -1$ and $x = 0$. Without using the table of values sketch the curve given by $y = x^3 - 6x^2 + 9x + 1$.

$f(x) = x^3 - 6x^2 + 9x + 1$.

$x = -1$: $f(-1) = -1 - 6 - 9 + 1 = -15$ (negative).

$x = 0$: $f(0) = 1$ (positive).

Sign change $\rightarrow$ root exists.

Sketch: Cubic, y-intercept (0,1), roots near -1, 0, 4 (approx).

Answer (a): Root exists between -1 and 0. Sketch: cubic, y-intercept at 1.

(b) Use Taylor's theorem to expand $(x + h)^{(1/2)}$ in ascending powers of h up to the term containing h^3. Hence obtain the value of $\sqrt{10}$ giving your answer correct to five decimal places.

$f(x) = x^{(1/2)}$, $x = 9$, $h = 1$ ($\sqrt{10} = \sqrt{(9+1)}$).

$f'(x) = (1/2)x^{(-1/2)}$, $f''(x) = (1/2)(-1/2)x^{(-3/2)}$, $f'''(x) = (1/2)(-1/2)(-3/2)x^{(-5/2)}$.

At $x = 9$: $f(9) = 3$, $f'(9) = 1/6$, $f''(9) = -1/108$, $f'''(9) = 5/1944$.

Taylor: $f(9+h) = 3 + (1/6)h - (1/108)h^2 + (5/1944)h^3$.

$h = 1$: $\sqrt{10} \approx 3 + 1/6 - 1/108 + 5/1944 \approx 3.15998$.

Answer (b): $\sqrt{10} \approx 3.15998$