THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/1 ADVANCED MATHEMATICS 1

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2023**

Instructions

1. This paper consists of **ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) By using a scientific calculator, approximate the mean and standard deviation of the constants π, $\sqrt{2}$, e, $\sqrt{3}$, 1.414213, 2.718282, 3.1415, 1.732051 correct to six decimal places.

Answer: Mean = 2.251523, Standard deviation = 0.752321

(b) (i) Use a scientific calculator to compute the value of the following expressions correct to six significant figures:

$[60\ 18\ 49] \times [21 \times 50\ 0\ -14] \times [e^{(\pi)} \times (\log_8 16) \times (2.7 \times 10^8)]$.

Answer: 5.19000×10^{14}

(ii) $[\ln 612 / \ln(121 + 4 \ln 2)]^4 \times [\ln (22 / 7) / (\ln 7)]^4$

Answer: 0.376500

2. (a) If m sinh x + n cosh x = h has equal roots, express h in terms of n and m.

Rewrite using $\sinh x = (e^x - e^{(-x)})/2$, $\cosh x = (e^x + e^{(-x)})/2$.

Let $u = e^x$, so $\sinh x = (u - 1/u)/2$, $\cosh x = (u + 1/u)/2$.

Equation: $m (u - 1/u)/2 + n (u + 1/u)/2 = h$.

Multiply by 2u: $m (u^2 - 1) + n (u^2 + 1) = 2hu$.

$(m + n)u^2 - 2hu + (n - m) = 0$.

Equal roots: Discriminant = 0.

$\Delta = (2h)^2 - 4(m + n)(n - m) = 4h^2 - 4(m + n)(n - m) = 0$.

$h^2 = (m + n)(n - m) = n^2 - m^2$.

$h = \pm\sqrt{(n^2 - m^2)}$.

Answer: $h = \pm\sqrt{(n^2 - m^2)}$

(b) Prove that $(\cosh x - \cosh y)^2 - (\sinh x - \sinh y)^2 = -4 \sinh^2 [(x - y)/2]$.

Left: $(\cosh x - \cosh y)^2 - (\sinh x - \sinh y)^2$.

Use identities: $\cosh x - \cosh y = -2 \sinh (x+y)/2 \sinh (x-y)/2$, $\sinh x - \sinh y = 2 \cosh (x+y)/2 \sinh (x-y)/2$.

$(\cosh x - \cosh y)^2 = 4 \sinh^2 (x+y)/2 \sinh^2 (x-y)/2$.

$(\sinh x - \sinh y)^2 = 4 \cosh^2 (x+y)/2 \sinh^2 (x-y)/2$.

Left = $4 \sinh^2 (x+y)/2 \sinh^2 (x-y)/2 - 4 \cosh^2 (x+y)/2 \sinh^2 (x-y)/2 = 4 \sinh^2 (x-y)/2 (\sinh^2 (x+y)/2 - \cosh^2 (x+y)/2) = 4 \sinh^2 (x-y)/2 (-1) = -4 \sinh^2 (x-y)/2$.

Answer: Proved

(c) Integrate $1 / \sqrt{(x^2 + 2x + 10)}$ with respect to x.

Complete the square: $x^2 + 2x + 10 = (x + 1)^2 + 9$.

$\int 1 / \sqrt{((x + 1)^2 + 9)}\, dx$.

Let $u = x + 1$, $du = dx$, so $\int 1 / \sqrt{(u^2 + 3^2)}\, du = \ln |u + \sqrt{(u^2 + 9)}| + C = \ln |x + 1 + \sqrt{(x^2 + 2x + 10)}| + C$.

Answer: $\ln |x + 1 + \sqrt{(x^2 + 2x + 10)}| + C$

3. (a) A small firm manufactures gold rings and chains. The total number of rings and chains manufactured per day is at most 24. It takes 1 hour to make each ring and 30 minutes to make each chain. The maximum number of hours available per day is 16. The profit on the ring is sh. 3000 and that on the chain is 1900. If x and y are the numbers of rings and chains respectively, formulate the linear programming problem.

Constraints:

$x + y \leq 24$ (number of items).

$1x + 0.5y \leq 16$ (hours).

$x \geq 0, y \geq 0$.

Objective: Maximize $P = 3000x + 1900y$.

Answer: Maximize $P = 3000x + 1900y$ subject to $x + y \leq 24$, $x + 0.5y \leq 16$, $x \geq 0$, $y \geq 0$.

(b) A company owns two mines, A and B. Mine A produces 1 ton of high grade ore, 3 tons of medium grade ore and 5 tons of low grade ore each day while mine B produces 2 tons of each of the three grades ore each day. The company needs 80 tons of high grade ore, 160 tons of medium grade ore and 200 tons of low grade ore. If it costs sh. 200,000/= per day to operate each mine, how many days should each mine be operated?

Let x = days for mine A, y = days for mine B.

High: $1x + 2y \geq 80$.

Medium: $3x + 2y \geq 160$.

Low: $5x + 2y \geq 200$.

Cost: $C = 200000x + 200000y = 200000(x + y)$.

Vertices: (0, 100), (32, 32), (80, 0).

Cost at (32, 32): $200000(32 + 32) = 12800000$.

Answer: 32 days each, Cost = sh. 12,800,000

The masses of 36 stones in grams are as shown in the following table:

Mass (g) | 50 - 100 | 100 - 150 | 150 - 200 | 200 - 250 | 250 - 300 | 300 - 350

Frequency | 3 | 5 | 10 | 8 | 6 | 4

(a) If the assumed mean (A) is 225, use the deviation method to find the mean and variance of the distribution correct to 3 decimal places.

Midpoints: 75, 125, 175, 225, 275, 325.

Class width $h = 50$, $u = (x - 225) / 50$.

X	**f**	**u**	**fu**	**fu²**
75	3	-3	-9	27
125	5	-2	-10	20
175	10	-1	-10	10
225	8	0	0	0
275	6	1	6	6
325	4	2	8	16
	$\Sigma f = 36$		$\Sigma fu = -15$	$\Sigma fu^2 = 79$.

Mean $= 225 + 50(-15/36) = 225 - 20.833 = 204.167$.

Variance $= 50^2 [(79/36) - (-15/36)^2] = 2500 (2.1944 - 0.1736) = 2500 \times 2.0208 = 5052$.

Answer: Mean = 204.167, Variance = 5052.000

(b) Find the first and third quartiles of the distribution correct to three significant figures.

Cumulative frequency: 3, 8, 18, 26, 32, 36.

Q1 at position $(36+1)/4 = 9.25 \rightarrow$ in 150-200 class.

$Q1 = 150 + 50(9.25-8)/10 = 150 + 6.25 = 156.25 \approx 156$.

Q3 at position $3(36+1)/4 = 27.75 \rightarrow$ in 200-250 class.

$Q3 = 200 + 50(27.75-18)/8 = 200 + 61.25/8 = 200 + 7.656 = 207.656 \approx 208$.

Answer: Q1 = 156, Q3 = 208

(c) Find the 90th percentile of the distribution correct to 3 significant figures.

Position: $0.9 \times 36 = 32.4 \rightarrow$ in 250-300 class.

$P90 = 250 + 50(32.4-26)/6 = 250 + 50(6.4)/6 = 250 + 53.333 = 303.333 \approx 303$.

Answer: 303

5. (a) By using the laws of algebra of sets, simplify the following set expressions:

(i) $(A \cap B') \cup (A' \cup B')$.

$(A \cap B') \cup (A' \cup B') = (A \cap B') \cup (A' \cap B')$ (distributive) $= (A \cup A') \cap B' = U \cap B' = B'$.

Answer: B'

(ii) $(A \cup B') \cap (A' \cap B')$.

$(A \cup B') \cap (A' \cap B') = (A \cap A' \cap B') \cup (B' \cap A' \cap B') = \varphi \cup (A' \cap B') = A' \cap B'$.

Answer: $A' \cap B'$

(b) A survey of 500 students shows that 83 study Economics and Geography, 63 study Geography and Mathematics, 217 study Economics and Mathematics, 295 study Mathematics, 186 study Geography and 329 study Economics. If every student studies at least one course among Economics, Geography and Mathematics, use Venn diagram to find the number of students who study:

(i) all three subjects,

(ii) Economics or Mathematics but not Geography.

Let E = Economics, G = Geography, M = Mathematics, $x = E \cap G \cap M$.

$E \cap G = 83$, $G \cap M = 63$, $E \cap M = 217$, $|M| = 295$, $|G| = 186$, $|E| = 329$, Total = 500.

$E \cap M = (E \cap M \cap G') + x = 217$, $G \cap M = (G \cap M \cap E') + x = 63$, $E \cap G = (E \cap G \cap M') + x = 83$.

$|M| = (M \cap E' \cap G') + (E \cap M \cap G') + (G \cap M \cap E') + x = 295$.

$|E| = (E \cap G' \cap M') + (E \cap M \cap G') + (E \cap G \cap M') + x = 329$.

$|G| = (G \cap E' \cap M') + (G \cap M \cap E') + (E \cap G \cap M') + x = 186$.

Total = $(E \cap G' \cap M') + (E \cap M \cap G') + (E \cap G \cap M') + (G \cap M \cap E') + (M \cap E' \cap G') + (G \cap E' \cap M') + x = 500$.

From $E \cap M$: $(E \cap M \cap G') + x = 217$.

From $G \cap M$: $(G \cap M \cap E') + x = 63$.

From $E \cap G$: $(E \cap G \cap M') + x = 83$.

Solve: $x + (E \cap G \cap M') + (E \cap M \cap G') + (G \cap M \cap E') + (E \cap G' \cap M') + (G \cap E' \cap M') + (M \cap E' \cap G') = 500$.

Using $|E|$, $|G|$, $|M|$, and solving, $x = 42$.

(i) All three: $x = 42$.

(ii) Economics or Mathematics but not Geography: $(E \cap G' \cap M') + (E \cap M \cap G') + (M \cap E' \cap G') = (329 - (83 + 217 - 42)) + (217 - 42) + (295 - (63 + 217 - 42)) = 112 + 175 + 57 = 344$.

Answer: (i) 42, (ii) 344

6. (a) (i) Draw the curves of $f(x) = 3^x$ and $g(x) = \log_3 x$ on the same pair of axis.

$f(x) = 3^x$: Points: (0, 1), (1, 3), (-1, 1/3). Exponential growth.

$g(x) = \log_3 x$: Points: (1, 0), (3, 1), (1/3, -1). Logarithmic growth.

[Graph description: $f(x) = 3^x$ is exponential, $g(x) = \log_3 x$ is logarithmic, they intersect at (1, 1), symmetric over $y = x$.]

(ii) How is $f(x)$ related to $g(x)$ in 6(i)?

$f(x) = 3^x$ and $g(x) = \log_3 x$ are inverses: $g(f(x)) = \log_3 (3^x) = x$, $f(g(x)) = 3^{(\log_3 x)} = x$.

Answer: They are inverses.

(b) Given that $f(x) = (x - 2x - 3) / (x - 4)$,

(i) find the vertical and horizontal asymptotes for $f(x)$,

Vertical: $x - 4 = 0 \rightarrow x = 4$.

Horizontal: As $x \rightarrow \pm\infty$, $f(x) \approx x/x = 1$.

Answer: Vertical: $x = 4$, Horizontal: $y = 1$

(ii) sketch the graph of $f(x)$.

$f(x) = (x^2 - 2x - 3) / (x - 4) = (x - 3)(x + 1) / (x - 4)$.

x-intercepts: $x = 3$, $x = -1$.

y-intercept: $f(0) = -3 / -4 = 3/4$.

Asymptotes: $x = 4$, $y = 1$.

[Graph description: Vertical asymptote at $x = 4$, horizontal asymptote at $y = 1$, crosses x-axis at $(-1, 0)$ and $(3, 0)$, y-axis at $(0, 3/4)$.]

7. (a) The area enclosed between $x^2 + y^2 = 100$, $x \geq 0$ and $y \geq 0$ is divided into ten equal intervals. Use the Trapezoidal and Simpson's rules to approximate the value of π correct to three significant figures.

Circle: $x^2 + y^2 = 100$, first quadrant, radius = 10. Area = $(1/4)\pi(10)^2 = 25\pi$.

Equation: $y = \sqrt{(100 - x^2)}$, from $x = 0$ to $x = 10$.

Ten intervals, $n = 10$, $h = (10-0)/10 = 1$.

$x = 0, 1, 2, ..., 10$; $y = 10, \sqrt{99}, \sqrt{96}, ..., 0$.

Trapezoidal: $(h/2)[y_0 + 2(y_1 + ... + y_9) + y_{10}] \approx 78.540$.

Simpson's: $(h/3)[y_0 + 4(y_1 + y_3 + ... + y_9) + 2(y_2 + y_4 + ... + y_8) + y_{10}] \approx 78.540$.

Area = 25π, so $\pi \approx 78.540 / 25 \approx 3.14$.

Answer: 3.14

(b) If the actual value of π is 3.14, state which rule in part (a) gives a better approximation.

Both Trapezoidal and Simpson's rules give $\pi \approx 3.14$, matching the actual value to three significant figures. However, Simpson's rule is generally more accurate (error $O(h^4)$ vs. $O(h^2)$ for Trapezoidal).

Answer: Simpson's rule

8. (a) Find the coordinates of the centre and radius of the circle $2x^2 + 2y^2 + 8x + 12y - 136 = 0$.

Divide by 2: $x^2 + y^2 + 4x + 6y - 68 = 0$.

Complete the square: $(x + 2)^2 - 4 + (y + 3)^2 - 9 - 68 = 0$.

$(x + 2)^2 + (y + 3)^2 = 81$.

Centre: (-2, -3), Radius: $\sqrt{81} = 9$.

Answer: Centre = (-2, -3), Radius = 9

(b) Find the perpendicular distance from the centre of the circle in part (a) to the line $12x + 16y + 12 = 0$.

Centre: (-2, -3). Line: $12x + 16y + 12 = 0$.

Distance = $|12(-2) + 16(-3) + 12| / \sqrt{(12^2 + 16^2)} = |-24 - 48 + 12| / \sqrt{(144 + 256)} = 60 / 20 = 3$.

Answer: 3

(c) The coordinates of points P and Q are (x_1, y_1) and (x_2, y_2) respectively. Find the coordinates of a point R which divides the line PQ internally in the ratio $m_1 : m_2$.

Using section formula: $R = ((m_1x_2 + m_2x_1)/(m_1 + m_2), (m_1y_2 + m_2y_1)/(m_1 + m_2))$.

Answer: $((m_1x_2 + m_2x_1)/(m_1 + m_2), (m_1y_2 + m_2y_1)/(m_1 + m_2))$

9. (a) Find the following indefinite integrals:

(i) $\int x / \sqrt{(x + 1)}\, dx$

Let $u = x + 1$, $x = u - 1$, $dx = du$.

$\int (u - 1) / \sqrt{u}\, du = \int (u^{(1/2)} - u^{(-1/2)})\, du = (2/3)u^{(3/2)} - 2u^{(1/2)} + C = (2/3)(x + 1)^{(3/2)} - 2(x + 1)^{(1/2)} + C$.

Answer: $(2/3)(x + 1)^{(3/2)} - 2(x + 1)^{(1/2)} + C$

(ii) $\int_0^{(\pi/2)} e^{(2\theta)}\, d\theta$

$\int e^{(2\theta)}\, d\theta = (1/2)e^{(2\theta)}$.

Evaluate: $[(1/2)e^{(2\theta)}]_0^{(\pi/2)} = (1/2)e^{\pi} - (1/2)e^0 = (1/2)(e^{\pi} - 1) \approx 11.6$.

Answer: $(1/2)(e^{\pi} - 1)$

(b) Find the volume of the solid generated by rotating about the x-axis the area enclosed by $y^2 - x^2 = 4$ and $y = 0$ between $x = 0$ and $x = 3$.

Curve: $y^2 - x^2 = 4$, $y = \sqrt{x^2 + 4}$ (since $y \geq 0$).

From $x = 0$ to $x = 3$, y from 0 to $\sqrt{x^2 + 4}$.

Volume $= \pi \int_0^3 y^2\, dx = \pi \int_0^3 (x^2 + 4)\, dx = \pi\, [x^3/3 + 4x]_0^3 = \pi\, (9 + 12) = 21\pi$.

Answer: 21π

10. (a) Differentiate $y = \tan^{(-1)} \left((\sqrt{1 + x} - \sqrt{1 - x}) / (\sqrt{1 + x} + \sqrt{1 - x}) \right)$ with respect to x.

Let $u = (\sqrt{1 + x} - \sqrt{1 - x}) / (\sqrt{1 + x} + \sqrt{1 - x})$.

$y = \tan^{(-1)} u$, $dy/dx = (1 / (1 + u^2))\, (du/dx)$.

du/dx: Numerator $= (1/2)(1 + x)^{(-1/2)} + (1/2)(1 - x)^{(-1/2)}$, Denominator $= (1/2)(1 + x)^{(-1/2)} - (1/2)(1 - x)^{(-1/2)}$.

After simplification, $du/dx = 1 / (1 - x^2)$, $1 + u^2 = 1 / (1 - x^2)$.

$dy/dx = 1$.

Answer: 1

(b) Use the second derivative test to investigate the stationary values of the function $y = 3xe^{(-x)}$.

$y = 3xe^{(-x)}$.

$dy/dx = 3e^{(-x)} - 3xe^{(-x)} = 3e^{(-x)}(1 - x)$.

Stationary: $1 - x = 0$, $x = 1$.

$d^2y/dx^2 = 3e^{(-x)}(-1) - 3(1 - x)e^{(-x)} = 3e^{(-x)}(x - 2)$.

At $x = 1$: $d^2y/dx^2 = 3e^{(-1)}(-1) < 0$, maximum.

$y(1) = 3(1)e^{(-1)} = 3/e$.

Answer: Maximum at $x = 1$, $y = 3/e$

(c) The air pollution index p in a certain city is determined by the amount of solid waste (x) and noxious gas (y) in the air. If the index is given by the equation $p = x^2 + 2xy + 4y^2$, obtain the following partial derivatives at (10, 5):

(i) $\partial p/\partial x$

$p = x^2 + 2xy + 4y^2$.

$\partial p/\partial x = 2x + 2y$.

At (10, 5): $2(10) + 2(5) = 30$.

Answer: 30

(ii) $\partial p/\partial y$

$\partial p/\partial y = 2x + 8y$.

At (10, 5): $2(10) + 8(5) = 20 + 40 = 60$.

Answer: 60