THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/1 ADVANCED MATHEMATICS 1

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2024**

Instructions

1. This paper consists of **ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Compute the following definite integrals correct to four decimal places using a non-programmable scientific calculator:

(i) $\int_0^1 (\cos\theta / (1 + \sin^2\theta))\, d\theta$

Answer: 0.6992

(ii) $\int_2^3 (2x - 3) / \sqrt{(2x - x^2)}\, dx$

Answer: 0.4292

(b) Calculate mean and standard deviation of π, $\sqrt{2}$, e, $\sqrt{3}$, 1.414213, 2.718282, 3.1415, 1.732051 to six decimal places.

Answer: Mean = 2.251523, Standard deviation = 0.752321

2. (a) Evaluate $\int_0^1 x \sinh 3x\, dx$ using integration by parts, correct to three decimal places.

$u = x$, $dv = \sinh 3x\, dx \rightarrow v = (1/3)\cosh 3x$.

$\int x \sinh 3x\, dx = [x\, (1/3)\cosh 3x]_0^1 - (1/3)\int_0^1 \cosh 3x\, dx = (1/3)\cosh 3 - (1/3)(1/3)(\sinh 3 - 0)$.

$\sinh 3 \approx 10.01785$, $\cosh 3 \approx 10.06765$.

Value $\approx (1/3)(10.06765) - (1/9)(10.01785) \approx 2.243$.

Answer: 2.243

(b) If $\cosh x = 1 + 4 \sinh x$, find x.

Let $y = e^x$. Then: $(y + 1/y)/2 = 1 + 4(y - 1/y)/2$.

Simplify: $3y^2 + 2y - 5 = 0 \rightarrow y = 1$ (discard $y = -5/3$).

So, $x = 0$. Verify: $\cosh 0 = 1$, $\sinh 0 = 0$, $1 + 4(0) = 1$.

Answer: $x = 0$

(c) Prove $\cosh^2 x + \sinh^2 x = 1 / \operatorname{sech} 2x$.

Left: $\cosh^2 x + \sinh^2 x = (e^{(2x)} + e^{(-2x)})/2$.

Right: 1 / sech 2x = cosh 2x = (e^(2x) + e^(-2x))/2.

Both equal, so proved.

3. Mr. Mashauri needs 10, 12, 12 units of A, B, C. Liquid: 5, 2, 1 units at Tsh.3,000/jar. Dry: 1, 2, 4 units at Tsh.2,000/carton. Minimize cost.

x = jars, y = cartons. Constraints: $5x + y \geq 10$, $2x + 2y \geq 12$, $x + 4y \geq 12$.

Cost: $C = 3000x + 2000y$.

Solve: $x = 1$, $y = 5$ (cost = 3000 + 10000 = 13000).

Answer: 1 jar, 5 cartons, Cost = Tsh.13,000

4. (a) Mean of 200 numbers was 50. Incorrect: 92, 8. Correct: 192, 88. Find correct mean.

Original sum = $200 \times 50 = 10000$.

New sum = $10000 - (92 + 8) + (192 + 88) = 10180$.

Mean = $10180 / 200 = 50.9$.

Answer: 50.9

(b) Use coding method (A = 21) for mean and standard deviation:

Marks: 0-6, 6-12, 12-18, 18-24, 24-30, 30-36, 36-42.

Frequency: 2, 3, 5, 10, 3, 5, 2.

Midpoints: 3, 9, 15, 21, 27, 33, 39.

$u = (x - 21)/6$. $\Sigma fu = 2$, $\Sigma fu^2 = 76$, $\Sigma f = 30$.

Mean = $21 + 6(2/30) = 21.4$.

$s^2 = (76/30) - (2/30)^2 (6^2)(30/29) \approx 94.138$, $s \approx 9.702$.

Answer: Mean = 21.4, Standard deviation ≈ 9.702

5. (a) By using the laws of algebra of sets, show that $(A \cup B)' \cap B = \varphi$.

The complement $(A \cup B)'$ is the set of elements not in $A \cup B$, so $(A \cup B)' = A' \cap B'$ (De Morgan's Law).

Then, $(A \cup B)' \cap B = (A' \cap B') \cap B$.

By associativity, this is $A' \cap (B' \cap B)$.

Since $B' \cap B = \varphi$ (an element cannot be in both B and its complement), we have:

$(A' \cap B') \cap B = A' \cap \varphi = \varphi$.

Answer: $(A \cup B)' \cap B = \varphi$

(b) Use the appropriate laws to simplify $[(A - B) - B] - [(A - B) \cap B]$.

First, simplify $(A - B) - B$:

$A - B = A \cap B'$, so $(A - B) - B = (A \cap B') - B = (A \cap B') \cap B' = A \cap B' \cap B' = A \cap B'$.

Next, $(A - B) \cap B = (A \cap B') \cap B = A \cap (B' \cap B) = A \cap \varphi = \varphi$.

So, $[(A - B) - B] - [(A - B) \cap B] = (A \cap B') - \varphi = A \cap B'$.

Answer: $A \cap B'$

(c) During the awarding day, 20 students received awards for academic excellence only, 30 received awards for generosity only and 35 students received awards for smartness only. Also, 10 students received awards for generosity and academic excellence but not smartness, 60 received awards for smartness and 55 students received awards for generosity. If the number of students who received awards for smartness and academic excellence is equal to the number of students who received awards for generosity and smartness, use Venn diagram to find:

(i) the number of students who received awards for academic excellence.

(ii) the total number of students.

Let:

A = academic excellence, G = generosity, S = smartness.

A only = 20, G only = 30, S only = 35, $G \cap A \cap S' = 10$, $|S| = 60$, $|G| = 55$.

$|S \cap A| = |S \cap G|$ (given). Let $|S \cap A| = |S \cap G| = x$.

Venn diagram regions:

$A \cap G \cap S = y$ (all three).

$A \cap S \cap G' = x - y$, $G \cap S \cap A' = x - y$, $A \cap G \cap S' = 10$.

For G: $30 + 10 + (x - y) + y = 55 \rightarrow 40 + x = 55 \rightarrow x = 15$.

For S: $35 + (x - y) + (x - y) + y = 60 \rightarrow 35 + 2(x - y) + y = 60 \rightarrow 35 + 2(15 - y) + y = 60 \rightarrow 35 + 30 - 2y + y = 60 \rightarrow 65 - y = 60 \rightarrow y = 5$.

So, $x = 15$, $y = 5$.

(i) Academic excellence: $20 + 10 + (x - y) + y = 20 + 10 + (15 - 5) + 5 = 20 + 10 + 10 + 5 = 45$.

(ii) Total students: $20 + 30 + 35 + 10 + (x - y) + (x - y) + y = 20 + 30 + 35 + 10 + 10 + 10 + 5 = 120$.

Answer: (i) 45, (ii) 120

6. (a) If $f(x) = x^2 - 12x - 7$, fill the following table with the missing values of $f(x)$.

x | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4

f(x) | | | | | | | | | |

$f(x) = x^2 - 12x - 7$:

$x = -4$: $(-4)^2 - 12(-4) - 7 = 16 + 48 - 7 = 57$

$x = -3$: $(-3)^2 - 12(-3) - 7 = 9 + 36 - 7 = 38$

$x = -2$: $(-2)^2 - 12(-2) - 7 = 4 + 24 - 7 = 21$

$x = -1$: $(-1)^2 - 12(-1) - 7 = 1 + 12 - 7 = 6$

$x = 0$: $(0)^2 - 12(0) - 7 = -7$

$x = 1$: $(1)^2 - 12(1) - 7 = 1 - 12 - 7 = -18$

$x = 2$: $(2)^2 - 12(2) - 7 = 4 - 24 - 7 = -27$

$x = 3$: $(3)^2 - 12(3) - 7 = 9 - 36 - 7 = -34$

$x = 4$: $(4)^2 - 12(4) - 7 = 16 - 48 - 7 = -39$

Answer: f(x) | 57 | 38 | 21 | 6 | -7 | -18 | -27 | -34 | -39

(b) Hence draw the graph of $f(x)$.

Using the table from (a), plot points: (-4, 57), (-3, 38), (-2, 21), (-1, 6), (0, -7), (1, -18), (2, -27), (3, -34), (4, -39).

The function $f(x) = x^2 - 12x - 7$ is a parabola opening upwards (since coefficient of x^2 is positive).

Vertex: $x = -b/(2a) = 12/(2\cdot1) = 6$, $f(6) = 6^2 - 12(6) - 7 = 36 - 72 - 7 = -43$.

The graph crosses the x-axis where $f(x) = 0$: $x^2 - 12x - 7 = 0 \rightarrow x = [12 \pm \sqrt{(144 + 28)}]/2 = [12 \pm \sqrt{172}]/2 \approx 12.55, -0.55$.

[Graph description: Parabola with vertex at (6, -43), x-intercepts at ≈(-0.55, 0) and ≈(12.55, 0), y-intercept at (0, -7).]

(c) If $f(x) = 2x^2 - x - 10$:

(i) Find the asymptotes.

$f(x) = 2x^2 - x - 10$ is a quadratic, so no vertical or horizontal asymptotes (degree of numerator = 2, no denominator).

Answer: No asymptotes

(ii) Sketch the graph of $f(x)$.

$f(x) = 2x^2 - x - 10$. Vertex: $x = -b/(2a) = 1/(2\cdot2) = 0.25$, $f(0.25) = 2(0.25)^2 - 0.25 - 10 = 0.125 - 0.25 - 10 = -10.125$.

x-intercepts: $2x^2 - x - 10 = 0 \rightarrow x = [1 \pm \sqrt{(1 + 80)}]/4 = [1 \pm 9]/4 \rightarrow x = 2.5, -2$.

y-intercept: $f(0) = -10$.

[Graph description: Parabola opening upwards, vertex at (0.25, -10.125), x-intercepts at (-2, 0) and (2.5, 0), y-intercept at (0, -10).]

(iii) State the domain and range of $f(x)$.

Domain: All real numbers (polynomial).

Range: Minimum value at vertex = -10.125, so range is $[-10.125, \infty)$.

Answer: Domain: $(-\infty, \infty)$, Range: $[-10.125, \infty)$

7. (a) State two limitations of the Newton-Raphson formula.

Requires the derivative $f'(x)$ to be computable and non-zero at the root.

May fail to converge if the initial guess is far from the root or if the function has a flat slope near the root.

Answer: 1. Needs $f'(x) \neq 0$ and computable. 2. May not converge with poor initial guess.

(b) Use the Newton-Raphson formula to show that the kth root of a positive number A is given by $x_{n+1} = (1/k) [(k-1) x_n + (A / x_n^{k-1})]$.

Let $f(x) = x^k - A$. We want $f(x) = 0$, so $x^k = A$.

Newton-Raphson: $x_{(n+1)} = x_n - f(x_n)/f'(x_n)$.

$f(x) = x^k - A$, $f'(x) = k x^{(k-1)}$.

$x_{(n+1)} = x_n - (x_n^k - A)/(k x_n^{(k-1)}) = (k x_n^k - x_n^k + A)/(k x_n^{(k-1)}) = ((k-1) x_n^k + A) / (k x_n^{(k-1)}) = (1/k) [(k-1) x_n + A / x_n^{(k-1)}]$.

Answer: $x_{(n+1)} = (1/k) [(k-1) x_n + A / x_n^{(k-1)}]$

(c) Use the Trapezoidal rule with five ordinates to approximate the value of $\int_0^1 (1 + x^2)^{1/2}\, dx$ correct to three decimal places.

Five ordinates means 4 intervals ($n = 4$). Limits: 0 to 1, $h = (1-0)/4 = 0.25$.

$x = 0, 0.25, 0.5, 0.75, 1$.

$f(x) = (1 + x^2)^{1/2}$: $f(0) = 1$, $f(0.25) = (1 + 0.0625)^{1/2} = 1.0308$, $f(0.5) = (1 + 0.25)^{1/2} = 1.1180$, $f(0.75) = (1 + 0.5625)^{1/2} = 1.25$, $f(1) = (1 + 1)^{1/2} = 1.4142$.

Trapezoidal rule: $(h/2) [f(0) + 2(f(0.25) + f(0.5) + f(0.75)) + f(1)]$

$= (0.25/2) [1 + 2(1.0308 + 1.1180 + 1.25) + 1.4142] = 0.125 [1 + 2(3.3988) + 1.4142] = 0.125 [1 + 6.7976 + 1.4142] = 0.125 \times 9.2118 \approx 1.151$.

Answer: 1.151

8. (a) What is the length of a tangent from (2, 2) to the circle $x^2 + y^2 + 6x - 2y = 0$.

Circle: $x^2 + y^2 + 6x - 2y = 0 \rightarrow (x + 3)^2 + (y - 1)^2 = 10$. Center: $(-3, 1)$, radius $= \sqrt{10}$.

Distance from (2, 2) to center: $\sqrt{(2 - (-3))^2 + (2 - 1)^2} = \sqrt{5^2 + 1^2} = \sqrt{26}$.

Length of tangent $= \sqrt{\text{distance}^2 - \text{radius}^2} = \sqrt{26 - 10} = \sqrt{16} = 4$.

Answer: 4

(b) Find the equation of the normal to the circle $x^2 + y^2 - 24x - 14y + 63 = 0$ at the point (9, 4).

Circle: $(x - 12)^2 + (y - 7)^2 = 100$. Center: (12, 7).

Slope of radius to (9, 4): $(4 - 7)/(9 - 12) = (-3)/(-3) = 1$.

Slope of normal (perpendicular to radius) $= -1$.

Normal through (9, 4): $y - 4 = -1(x - 9) \rightarrow y - 4 = -x + 9 \rightarrow x + y - 13 = 0$.

Answer: $x + y - 13 = 0$

(c) Find the distance of the point (3, 2) from the normal line in part (b) correct to two decimal places.

Normal: $x + y - 13 = 0$.

Distance from (3, 2): $|3 + 2 - 13| / \sqrt{(1^2 + 1^2)} = |-8| / \sqrt{2} = 8 / \sqrt{2} = 4\sqrt{2} \approx 5.66$.

Answer: 5.66

9. (a) Find $\int (5 / (x^2 + x - 6))\, dx$.

Factor: $x^2 + x - 6 = (x + 3)(x - 2)$.

Use partial fractions: $5 / ((x + 3)(x - 2)) = A / (x + 3) + B / (x - 2)$.

$5 = A(x - 2) + B(x + 3) \rightarrow x = 2: 5 = B(5) \rightarrow B = 1; x = -3: 5 = A(-5) \rightarrow A = -1$.

$\int [-1 / (x + 3) + 1 / (x - 2)]\, dx = -\ln|x + 3| + \ln|x - 2| + C = \ln| (x - 2) / (x + 3) | + C$.

Answer: $\ln| (x - 2) / (x + 3) | + C$

(b) Evaluate $\int_0^2 \sin^2 x\, dx$ correct to four decimal places.

Use identity: $\sin^2 x = (1 - \cos 2x) / 2$.

$\int (1 - \cos 2x) / 2\, dx = (1/2) [x - (1/2) \sin 2x]_0^2 = (1/2) [(2 - (1/2) \sin 4) - (0 - 0)] = 1 - (1/4) \sin 4$.

$\sin 4 \approx \sin(4 \times 180/\pi) \approx \sin(229.18°) \approx -0.7568$.

Value $\approx 1 - (1/4)(-0.7568) = 1 + 0.1892 = 1.1892$.

Answer: 1.1892

(c) Find the length of the curve given by $x = 2 \cos \theta$ and $y = 2 \sin \theta$ between $\theta = 0$ and $\theta = \pi/2$.

Curve is a circle: $x^2 + y^2 = 4$, radius = 2.

θ from 0 to $\pi/2$ is a quarter circle (arc length of full circle $= 2\pi r = 4\pi$).

Quarter arc length $= (1/4) \times 4\pi = \pi \approx 3.1416$.

Answer: $\pi \approx 3.1416$

10. (a) If $x^3 + y^3 + x = -2y$, find dy/dx at $(-1, 1)$.

Differentiate implicitly:

$3x^2 + 3y^2\, dy/dx + 1 = -2\, dy/dx$.

Rearrange: $3y^2\, dy/dx + 2\, dy/dx = -3x^2 - 1$.

$dy/dx (3y^2 + 2) = -3x^2 - 1$.

$dy/dx = (-3x^2 - 1) / (3y^2 + 2)$.

At (-1, 1): $dy/dx = (-3(-1)^2 - 1) / (3(1)^2 + 2) = (-3 - 1) / (3 + 2) = -4/5$.

Answer: -4/5

(b) An object starts from rest and moves a distance of $g = (1/8)t^4 + (1/2)t^2$ cm in t seconds. By using this information find:

(i) the velocity of the object after two seconds.

(ii) the initial acceleration of the object.

Distance: $g = (1/8)t^4 + (1/2)t^2$.

Velocity $v = dg/dt = (1/8)(4t^3) + (1/2)(2t) = (1/2)t^3 + t$.

(i) At $t = 2$: $v = (1/2)(2^3) + 2 = (1/2)(8) + 2 = 4 + 2 = 6$ cm/s.

Acceleration $a = dv/dt = (1/2)(3t^2) + 1 = (3/2)t^2 + 1$.

(ii) At $t = 0$ (initial): $a = (3/2)(0) + 1 = 1$ cm/s².

Answer: (i) 6 cm/s, (ii) 1 cm/s²

(c) Differentiate $y = \tan^{-1} ((a \sin x + b \cos x) / (a \cos x - b \sin x))$ with respect to x.

Let $u = (a \sin x + b \cos x) / (a \cos x - b \sin x)$.

$y = \tan^{-1} u$, so $dy/dx = (1 / (1 + u^2)) (du/dx)$.

Differentiate u using quotient rule:

Numerator: $a \sin x + b \cos x$, Denominator: $a \cos x - b \sin x$.

$du/dx = [(a \cos x - b \sin x)(a \cos x - b \sin x) - (a \sin x + b \cos x)(-a \sin x - b \cos x)] / (a \cos x - b \sin x)^2$.

Numerator simplifies: $(a^2 + b^2) / (a \cos x - b \sin x)^2$.

So, $du/dx = (a^2 + b^2) / (a \cos x - b \sin x)^2$.

$u^2 = (a \sin x + b \cos x)^2 / (a \cos x - b \sin x)^2$, $1 + u^2 = (a^2 + b^2) / (a \cos x - b \sin x)^2$.

$dy/dx = [1 / (1 + u^2)] (du/dx) = [(a \cos x - b \sin x)^2 / (a^2 + b^2)] [(a^2 + b^2) / (a \cos x - b \sin x)^2] = 1$.

Answer: 1