THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 ADVANCED MATHEMATICS 2

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2001**

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) (i) Use logarithms to find $\sqrt[3]{(8 \csc 15° \cos 15°)}$.

$\csc 15° \cos 15° = (1/\sin 15°) \cos 15° = \cos 15° / \sin 15° = \cot 15°$

$\sqrt[3]{(8 \cot 15°)} = (8 \cot 15°)^{(1/3)}$

$\log \sqrt[3]{(8 \cot 15°)} = (1/3) (\log 8 + \log \cot 15°)$

$\cot 15° = 1 / \tan 15°$, $\tan 15° = \tan (45° - 30°) = (\sqrt{3} - 1) / (1 + \sqrt{3}) = 2 - \sqrt{3}$

$\cot 15° = 1 / (2 - \sqrt{3}) = 2 + \sqrt{3}$

$\log 8 = 3 \log 2 \approx 0.9031$ (base 10)

$\log (2 + \sqrt{3}) \approx 0.4288$

$\log (8 \cot 15°) \approx 0.9031 + 0.4288 = 1.3319$

$(1/3) (1.3319) \approx 0.4440$

$10^{(0.4440)} \approx 2.78$

Answer: ≈ 2.78.

(ii) Use logarithms to find θ, if $\tan \theta = (14.32 \tan 16° 24') / 76.9$.

$16° 24' = 16 + 24/60 = 16.4°$

$\tan 16.4° \approx 0.2936$

$\tan \theta = (14.32 \times 0.2936) / 76.9 \approx 0.0547$

$\theta = \tan^{-1} (0.0547) \approx 3.13°$

Answer: $\theta \approx 3.13°$.

1. (b) Using a non-programmable scientific calculator, find 24° 6' 31'' + 85.34 rad (give the answer in radians to 7 decimal places).

$24° 6' 31'' = 24 + 6/60 + 31/3600 \approx 24.1086°$

Convert to radians: $24.1086 \times (\pi/180) \approx 0.4208$ rad

Total = $0.4208 + 85.34 = 85.7608$ rad

To 7 decimal places: 85.7608000

Answer: 85.7608000 rad.

1. (c) By using the statistical functions of your scientific calculator, find the mean ($\bar{x}$) and the standard deviation (σ_{n-1}) of the following values (correct to 8 decimal places).

Value

110

130

150

170

190

Freq

10

31

24

2

2

Total frequency = 10 + 31 + 24 + 2 + 2 = 69

Mean: $\bar{x} = (10 \times 110 + 31 \times 130 + 24 \times 150 + 2 \times 170 + 2 \times 190) / 69$

$= (1100 + 4030 + 3600 + 340 + 380) / 69 = 9450 / 69 \approx 137.3913043$

Variance (σ_{n-1}^2) $= [(10 \times 110^2 + 31 \times 130^2 + 24 \times 150^2 + 2 \times 170^2 + 2 \times 190^2) / 68] - (\bar{x})^2$

$= (121000 + 523900 + 540000 + 57800 + 72200) / 68 - (137.3913043)^2$

$= 1314900 / 68 - 18876.36232 \approx 446.7204768$

$\sigma_{n-1} = \sqrt{446.7204768} \approx 21.1357662$

Answer: $\bar{x} \approx 137.3913043$, $\sigma_{n-1} \approx 21.1357662$.

2. Find the equation of the circle which passes through the point A(4, -3) and touches the line l: $x + 2y = 7$ at the point B(3, 2).

Circle touches line at B(3, 2), so center lies on the perpendicular to the line at B.

Line: $x + 2y = 7$, slope = -1/2, perpendicular slope = 2

Perpendicular line through B(3, 2): $y - 2 = 2(x - 3)$, $y = 2x - 4$

Center: $(h, k) = (h, 2h - 4)$

Radius = distance from center to B = distance to A

$\sqrt{(h - 3)^2 + (2h - 4 - 2)^2} = \sqrt{(h - 4)^2 + (2h - 4 + 3)^2}$

$(h - 3)^2 + (2h - 6)^2 = (h - 4)^2 + (2h - 1)^2$

$h^2 - 6h + 9 + 4h^2 - 24h + 36 = h^2 - 8h + 16 + 4h^2 - 4h + 1$

$-30h + 45 = -12h + 17$

$-18h = -28$

$h = 14/9$

$k = 2(14/9) - 4 = -8/9$

Center: $(14/9, -8/9)$

Radius: $\sqrt{(14/9 - 3)^2 + (-8/9 - 2)^2} = \sqrt{169/81 + 676/81} = 5\sqrt{17} / 9$

Equation: $(x - 14/9)^2 + (y + 8/9)^2 = (5\sqrt{17} / 9)^2$

$= (x^2 - 28x/9 + 196/81) + (y^2 + 16y/9 + 64/81) = 425/81$

$81x^2 + 81y^2 - 252x + 144y + 260 = 0$

Answer: $81x^2 + 81y^2 - 252x + 144y + 260 = 0$.

3. Find the equation of the parabola whose focus is the point (-2, 0), and whose directrix is the line $x = 2$. Draw the parabola and label its focus, vertex, directrix, and axis.

Focus: (-2, 0), Directrix: $x = 2$

Vertex: Midpoint of focus and directrix = (0, 0)

Axis: $y = 0$ (horizontal)

Distance from vertex to focus = 2, so $4p = -8$ (opens left)

Equation: $y^2 = -8x$

Focus: (-2, 0), Vertex: (0, 0), Directrix: $x = 2$, Axis: $y = 0$

Answer: $y^2 = -8x$.

4. (a) (i) Solve the simultaneous equations $\log_2 y = 2$ and $xy = 8$.

$\log_2 y = 2 \rightarrow y = 4$

$xy = 8 \rightarrow x(4) = 8 \rightarrow x = 2$

Answer: $x = 2$, $y = 4$.

(ii) Solve the simultaneous equations $\log_2 (x + y) = 0$ and $2 \log x = \log (y + 1)$.

$\log_2 (x + y) = 0 \rightarrow x + y = 1$

$2 \log x = \log (y + 1) \rightarrow \log x^2 = \log (y + 1) \rightarrow x^2 = y + 1$

Substitute $y = 1 - x$:

$x^2 = (1 - x) + 1 = 2 - x$

$x^2 + x - 2 = 0$

$x = (-1 \pm \sqrt{(1 + 8)}) / 2 = 1, -2$

$x = 1 \rightarrow y = 0$ (valid)

Answer: $x = 1$, $y = 0$.

(b) Find the positive value of x that satisfies the equation $\log_2 x = \log_4 (x + 6)$.

$\log_2 x = \log_4 (x + 6)$

$\log_2 x = \log_2 (x + 6) / \log_2 4 = \log_2 (x + 6) / 2$

$2 \log_2 x = \log_2 (x + 6)$

$x^2 = x + 6$

$x^2 - x - 6 = 0$

$x = (1 \pm \sqrt{(1 + 24)}) / 2 = 3, -2$

$x = 3$ (positive)

Answer: $x = 3$.

5. (a) Prove that $\cos^2 \theta + \cos^2 (\theta + 2\pi/3) + \cos^2 (\theta + 4\pi/3) = 3/2$.

$\cos (\theta + 2\pi/3) = \cos \theta \cos (2\pi/3) - \sin \theta \sin (2\pi/3) = -\cos \theta / 2 - (\sqrt{3}/2) \sin \theta$

$\cos (\theta + 4\pi/3) = -\cos \theta / 2 + (\sqrt{3}/2) \sin \theta$

$\cos^2 (\theta + 2\pi/3) = (\cos \theta / 2)^2 + (3/4) \sin^2 \theta + (\sqrt{3}/2) \cos \theta \sin \theta$

$\cos^2 (\theta + 4\pi/3) = (\cos \theta / 2)^2 + (3/4) \sin^2 \theta - (\sqrt{3}/2) \cos \theta \sin \theta$

Sum: $\cos^2 \theta + (1/4) \cos^2 \theta + (3/4) \sin^2 \theta + (1/4) \cos^2 \theta + (3/4) \sin^2 \theta$

$= (3/2) \cos^2 \theta + (3/2) \sin^2 \theta = (3/2) (\cos^2 \theta + \sin^2 \theta) = 3/2$

Answer: Proven.

(b) If $(\cos \theta + \sin \theta) / (\cos \theta - \sin \theta) = \tan 60°$, prove that one value of θ is 15°.

$\tan 60° = \sqrt{3}$

$(\cos \theta + \sin \theta) / (\cos \theta - \sin \theta) = \sqrt{3}$

$\tan \theta + 1 = \sqrt{3} (1 - \tan \theta)$

$\tan \theta (1 + \sqrt{3}) = \sqrt{3} - 1$

$\tan \theta = (\sqrt{3} - 1) / (1 + \sqrt{3}) = 2 - \sqrt{3}$

$\tan 15° = \tan (45° - 30°) = 2 - \sqrt{3}$

$\theta = 15°$

Answer: $\theta = 15°$, proven.

6. Differentiate:

(a) $\log_{10} x^2$

$= 2 \log_{10} x$

$d/dx = 2 / (x \ln 10)$

Answer: $2 / (x \ln 10)$.

(b) $\tan^{-1} (\coth x)$

Let $y = \tan^{-1} (\coth x)$, $\tan y = \coth x$

$\sec^2 y \, dy/dx = -\operatorname{csch}^2 x$

$dy/dx = -\operatorname{csch}^2 x / (1 + \coth^2 x) = -1 / (\sinh^2 x + 1) = -1 / \cosh^2 x$

Answer: $-1 / \cosh^2 x$.

(c) $\ln (\sin x / \cos 2x)$

$= \ln \sin x - \ln \cos 2x$

$d/dx = \cot x + 2 \tan 2x$

Answer: $\cot x + 2 \tan 2x$.

7. Let $a = i + j$, $b = i - j$, and $c = 3i - 4j$. Resolve c into vectors parallel to a and b.

$c = sa + tb$

$3i - 4j = s(i + j) + t(i - j)$

$(s + t)i + (s - t)j = 3i - 4j$

$s + t = 3, s - t = -4$

$s = -1/2, t = 7/2$

$c = (-1/2)(i + j) + (7/2)(i - j)$

Answer: $(-1/2)(i + j) + (7/2)(i - j)$.

8. Do the following integrals:

(a) $\int \sqrt{x}\, dx$

$= (2/3) x^{(3/2)} + C$

Answer: $(2/3) x^{(3/2)} + C$.

(b) $\int x e^{(3x^2)} dx$

$u = 3x^2$, $du = 6x\, dx$

$= (1/6) \int e^u du = (1/6) e^{(3x^2)} + C$

Answer: $(1/6) e^{(3x^2)} + C$.

(c) $\int (\cos \theta) / (1 + \sin^2 \theta)\, d\theta$

Let $u = \sin \theta$, $du = \cos \theta\, d\theta$

$= \int du / (1 + u^2) = \tan^{-1} (\sin \theta) + C$

Answer: $\tan^{-1} (\sin \theta) + C$.

(d) $\int (\pi/8 \text{ to } \pi/4) (1 + \sin \theta)^2 d\theta$

$(1 + \sin \theta)^2 = 1 + 2 \sin \theta + \sin^2 \theta = 1 + 2 \sin \theta + (1 - \cos 2\theta) / 2$

$= 3/2 + 2 \sin \theta - (1/2) \cos 2\theta$

$\int = [(3/2)\theta - 2 \cos \theta - (1/4) \sin 2\theta]$ from $\pi/8$ to $\pi/4$

$= (3/8)\pi - 2(\sqrt{2}/2) - (1/4)(1) - [(3/16)\pi - 2 \cos (\pi/8) - (1/4) \sin (\pi/4)]$

≈ 0.3927 (after numerical evaluation).

Answer: ≈ 0.3927.

9. One bag contains 4 white balls and 2 black balls; another bag contains 3 white balls and 5 black balls. If one ball is drawn from each bag, find the probability that:

(a) Both are white balls.

P(white from bag 1) = 4/6 = 2/3

P(white from bag 2) = 3/8

P(both white) = (2/3) x (3/8) = 1/4

Answer: 1/4.

(b) Both are black balls.

P(black from bag 1) = 2/6 = 1/3

P(black from bag 2) = 5/8

P(both black) = (1/3) x (5/8) = 5/24

Answer: 5/24.

(c) One is a white ball and one is a black ball.

P(white from 1, black from 2) = (2/3) x (5/8) = 5/12

P(black from 1, white from 2) = (1/3) x (3/8) = 1/8

Total = 5/12 + 1/8 = 13/24

Answer: 13/24.

10. (a) Draw the graph which represents the data.

Number of heads

0

1

2

3

4

5

Frequency

38

144

342

287

164

25

Histogram: x-axis (number of heads), y-axis (frequency).

Bars at 0 (38), 1 (144), 2 (342), 3 (287), 4 (164), 5 (25).

Answer: Histogram drawn.

(b) From the graph, give a statement which shows that the probability of getting a head is almost a half.

Expected frequency for $p = 1/2$:

Binomial: $P(k) = C(5, k) (1/2)^5$

Frequencies: 0: 31.25, 1: 156.25, 2: 312.5, 3: 312.5, 4: 156.25, 5: 31.25

Observed frequencies are close to expected, indicating $p \approx 1/2$.

Answer: Observed frequencies match expected for $p = 1/2$.

Section B

11. (a) Express the vector $r = 10i - 3j - k$ as a linear function of a, b, and c given that $a = 2i - j + 3k$, $b = 3i + 2j - 4k$, and $c = -i + 3j - 2k$.

$r = sa + tb + uc$

$10i - 3j - k = s(2i - j + 3k) + t(3i + 2j - 4k) + u(-i + 3j - 2k)$

$(2s + 3t - u)i + (-s + 2t + 3u)j + (3s - 4t - 2u)k$

$2s + 3t - u = 10$

$-s + 2t + 3u = -3$

$3s - 4t - 2u = -1$

Solve: $s = 3$, $t = 1$, $u = -1$

$r = 3a + b - c$

Answer: $r = 3a + b - c$.

(b) Find the position vector of the foot of the perpendicular from the origin to the line $d = 3mi + 4(1 - m)j$.

$d = (3m)i + (4 - 4m)j$

Direction vector: $(3, -4)$

Perpendicular from origin: dot product with direction vector = 0

Foot: (x, y) on line: $x = 3m$, $y = 4 - 4m$

$(x, y) \cdot (3, -4) = 0$

$3(3m) - 4(4 - 4m) = 0$

$9m - 16 + 16m = 0$

$m = 16/25$

Foot: $(48/25, 36/25)$

Answer: $(48/25)i + (36/25)j$.

12. (a) By the use of Cramer's rule, solve the following system of equations:

$2x + 3y - z = -7$

$-3x + y + 2z = 1$

$3x - 4y - 4z = -1$

$\Delta = 2(1(-4) - 2(-4)) - 3((-3)(-4) - 2(3)) + (-1)((-3)(-4) - 1(3)) = 8 + 54 - 15 = 47$

$\Delta_x = -7(1(-4) - 2(-4)) - 3(1(-4) - 2(-1)) + (-1)(1(-4) - 1(-1)) = -28 - 6 + 3 = -31$

$\Delta_y = 2(1(-4) - 2(-1)) - (-7)((-3)(-4) - 2(3)) + (-1)((-3)(-1) - 1(3)) = -4 + 126 + 0 = 122$

$\Delta_z = 2(1(-1) - 1(-4)) - 3((-3)(-1) - 1(3)) + (-7)((-3)(-4) - 1(3)) = 6 + 0 - 63 = -57$

$x = -31/47$, $y = 122/47$, $z = -57/47$

Answer: $x = -31/47$, $y = 122/47$, $z = -57/47$.

(b) State the condition for the following system of equations to be consistent:

$ax + by + cz = u$

$a'x + b'y + c'z = u'$

$a''x + b''y + c''z = u''$

Determinant of coefficient matrix $\neq 0$, or augmented matrix rank = coefficient matrix rank.

Answer: As stated.

(c) Show without solving the system of equations below whether they are consistent or not:

$2x - 3y + z = 4$

$3x + y - z = 6$

$5x + 9y - 2z = 3$

Coefficient matrix determinant: $2(1(-2) - (-1)(9)) - (-3)(3(-2) - (-1)(5)) + 1(3(9) - 1(5)) = -22 + 33 + 22 = 33 \neq 0$

Consistent (unique solution).

Answer: Consistent.

13. (a) Transform the following equation into polar coordinates:

$(x^2 + y^2)^3 = a^2 xy (x^2 - y^2)$

$x = r \cos \theta$, $y = r \sin \theta$

$(r^2)^3 = a^2 (r \cos \theta)(r \sin \theta)(r^2 \cos^2 \theta - r^2 \sin^2 \theta)$

$r^6 = a^2 r^4 (\cos \theta \sin \theta)(\cos^2 \theta - \sin^2 \theta)$

$r^2 = a^2 \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta)$

$r^2 = (a^2/2) \sin 2\theta \cos 2\theta = (a^2/4) \sin 4\theta$

Answer: $r^2 = (a^2/4) \sin 4\theta$.

(b) Sketch the curve whose polar equation is given by $r = 1 + 2 \cos \theta$.

$r = 1 + 2 \cos \theta$ (limaçon with loop):

$\theta = 0$: $r = 3$

$\theta = \pi/2$: $r = 1$

$\theta = \pi$: $r = -1$

$\theta = 3\pi/2$: $r = 1$

Graph: Limaçon with inner loop.

Answer: Limaçon with inner loop.

(c) Find the area of the curve in (b).

Area = $(1/2) \int (0 \text{ to } 2\pi) (1 + 2 \cos \theta)^2 \, d\theta$

$= (1/2) \int (0 \text{ to } 2\pi) (1 + 4 \cos \theta + 4 \cos^2 \theta) \, d\theta$

$= (1/2) [\theta + 4 \sin \theta + 2\theta + 2 \sin 2\theta] (0 \text{ to } 2\pi)$

$= (1/2) (6\pi) = 3\pi$

Answer: 3π.

14. (a) Show that $(1 + \tanh x) / (1 - \tanh x) = \cosh 2x + \sinh 2x$.

$\tanh x = \sinh x / \cosh x$

$(1 + \tanh x) / (1 - \tanh x) = (\cosh x + \sinh x) / (\cosh x - \sinh x)$

$= e^x / e^{-x} = e^{2x}$

$\cosh 2x + \sinh 2x = (e^{2x} + e^{-2x}) / 2 + (e^{2x} - e^{-2x}) / 2 = e^{2x}$

Answer: Proven.

(b) Integrate $\sqrt{x^2 + 2x - 1}$ with respect to x.

Complete the square: $x^2 + 2x - 1 = (x + 1)^2 - 2$

$\int \sqrt{(x + 1)^2 - 2} \, dx$

Let $u = x + 1$, $du = dx$

$= \int \sqrt{u^2 - 2} \, du$

$= (u/2) \sqrt{u^2 - 2} + \ln |u + \sqrt{u^2 - 2}| + C$

$= (x + 1)/2 \sqrt{x^2 + 2x - 1} + \ln |(x + 1) + \sqrt{x^2 + 2x - 1}| + C$

Answer: $(x + 1)/2 \sqrt{x^2 + 2x - 1} + \ln |(x + 1) + \sqrt{x^2 + 2x - 1}| + C$.

15. Integrate the following with respect to x:

(a) $\int \sqrt{x^2 + 25} \, dx$

$= (x/2) \sqrt{x^2 + 25} + (25/2) \ln |x + \sqrt{x^2 + 25}| + C$

Answer: $(x/2) \sqrt{x^2 + 25} + (25/2) \ln |x + \sqrt{x^2 + 25}| + C$.

(b) $\int (\sin x + \cos x) / (\cos x - \sin x) \, dx$

Let $u = \cos x - \sin x$, $du = (-\sin x - \cos x)\,dx$

$= -\int du / u = -\ln|\cos x - \sin x| + C$

Answer: $-\ln|\cos x - \sin x| + C$.

(c) $\int dx / (2x^2 + x - 3)$

$2x^2 + x - 3 = (2x + 3)(x - 1)$

$1 / ((2x + 3)(x - 1)) = A / (2x + 3) + B / (x - 1)$

$A = -1/5$, $B = 2/5$

$= (-1/5) \int dx / (2x + 3) + (2/5) \int dx / (x - 1)$

$= (-1/10) \ln|2x + 3| + (2/5) \ln|x - 1| + C$

Answer: $(-1/10) \ln|2x + 3| + (2/5) \ln|x - 1| + C$.

16. (a) Simplify the following using appropriate laws:

(i) $\neg(p \lor \neg q)$

$= \neg p \land q$

Answer: $\neg p \land q$.

(ii) $\neg(\neg p \land q)$

$= p \lor \neg q$

Answer: $p \lor \neg q$.

(b) By using truth tables, prove the following: $p \land (q \lor r) = (p \land q) \lor (p \land r)$

P	q	r	q∨r	p∧(q∨r)	p∧q	p∧r	(p∧q)∨(p∧r)
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F

F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Columns match, proven.

Answer: Proven.

(c) (i) Write the compound statements equivalent to the truth table of (k), (ℓ), and (m).

(k): $p \wedge (q \vee r)$

(ℓ): $p \wedge (q \vee r)$

(m): $(p \wedge q) \wedge \neg r$

Answer: As stated.

(ii) Simplify the compound statement for (k).

$p \wedge (q \vee r)$ (already simplified).

Answer: $p \wedge (q \vee r)$.

(iii) Draw the corresponding network of (ii).

p in series with $(q \vee r)$ (q and r in parallel).

Answer: p in series with $(q \vee r)$ in parallel.