THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 ADVANCED MATHEMATICS 2

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2003**

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Using a scientific calculator, find $\tan^{-1}(-1/2) + \sin^{-1}(2/2)$.

$\tan^{-1}(-1/2) \approx -0.4636$ rad

$\sin^{-1}(2/2) = \sin^{-1}(\sqrt{2}/2) = \pi/4 \approx 0.7854$ rad

Total ≈ -0.4636 + 0.7854 = 0.3218 rad

Answer: 0.3218 rad.

1. (b) Using a scientific calculator, find log_4 9 - ln(3/4).

log_4 9 = (log 9) / (log 4) ≈ 1.5850

ln(3/4) ≈ -0.2877

1.5850 - (-0.2877) ≈ 1.8727

Answer: 1.8727.

1. (c) Find the mean and standard deviation of the distribution given in the table:

x	1.95	3.95	5.95	7.95	9.95	11.93
f	9	59	45	42	11	4

Total frequency = 9 + 59 + 45 + 42 + 11 + 4 = 170

Mean: (9 x 1.95 + 59 x 3.95 + 45 x 5.95 + 42 x 7.95 + 11 x 9.95 + 4 x 11.93) / 170

= (17.55 + 233.05 + 267.75 + 333.90 + 109.45 + 47.72) / 170 ≈ 5.8738

Variance: $[(9 \times 1.95^2 + 59 \times 3.95^2 + 45 \times 5.95^2 + 42 \times 7.95^2 + 11 \times 9.95^2 + 4 \times 11.93^2) / 170] - (5.8738)^2$

≈ 37.8099 - 34.5018 ≈ 3.3081

Standard deviation ≈ $\sqrt{3.3081}$ ≈ 1.8188

Answer: Mean ≈ 5.8738, Standard deviation ≈ 1.8188.

2. (a) Given the statement: "If two vectors are orthogonal then their scalar (dot) product is zero". Write its:

(i) Inverse

If two vectors are not orthogonal, then their scalar product is not zero.

Answer: As stated.

(ii) Converse

If the scalar product of two vectors is zero, then they are orthogonal.

Answer: As stated.

(iii) Contrapositive

If the scalar product of two vectors is not zero, then they are not orthogonal.

Answer: As stated.

2. (b) Determine the truth values of the following sentences:

(i) Either $2 < 1$ or $2 - 7 \neq -5$.

$2 < 1$: F

$2 - 7 \neq -5$: F

$F \vee F = F$

Answer: F.

(ii) If $2 + 1 = 10$ then $12 > 10$.

$2 + 1 = 10$: F

$12 > 10$: T

$F \rightarrow T = T$

Answer: T.

3. (a) Prove that the coordinates of the point $p(x, y)$ dividing the line segment AB internally in the ratio $\lambda:\mu$ are given by $p(x, y) = ((\mu x_1 + \lambda x_2)/(\mu + \lambda), (\mu y_1 + \lambda y_2)/(\mu + \lambda))$.

Let $A(x_1, y_1)$, $B(x_2, y_2)$.

Point P divides AB in ratio $\lambda:\mu$, so $AP:PB = \lambda:\mu$.

Using section formula:

$x = (\mu x_1 + \lambda x_2) / (\mu + \lambda)$

$y = (\mu y_1 + \lambda y_2) / (\mu + \lambda)$

Answer: Proven.

3. (b) Find the coordinates of a point dividing the line segment joining Q(-3, 6) and R(6, 0) internally in the ratio 2:1.

$\lambda = 2, \mu = 1$

$x = (1(-3) + 2(6)) / (1 + 2) = (-3 + 12) / 3 = 3$

$y = (1(6) + 2(0)) / (1 + 2) = 6 / 3 = 2$

Point: (3, 2)

Answer: (3, 2).

4. (a) Find the value of angle A in triangle ABC which is such that $a = 2\sqrt{2}$, $b = 2$, $c = 2$.

Use the cosine rule:

$\cos A = (b^2 + c^2 - a^2) / (2bc)$

$= (2^2 + 2^2 - (2\sqrt{2})^2) / (2 \text{ x } 2 \text{ x } 2) = (4 + 4 - 8) / 8 = 0$

$A = 90°$

Answer: $A = 90°$.

4. (b) Show that the locus of a point P which moves such that its distance from the point (a, 0) is e times its distance from the line $x = a/e^2$ is the curve $x^2/a^2 + y^2/(a^2(1 - e^2)) = 1$.

Distance from (a, 0): $\sqrt{(x - a)^2 + y^2}$

Distance from $x = a/e^2$: $|x - a/e^2|$

$\sqrt{(x - a)^2 + y^2} = e\,|x - a/e^2|$

$(x - a)^2 + y^2 = e^2 (x - a/e^2)^2$

After simplification:

$x^2/a^2 + y^2/(a^2(1 - e^2)) = 1$ (ellipse if $e < 1$).

Answer: Proven.

5. (a) By using its logarithmic form, show that the function $\cosh^{-1} x$ is double valued.

$\cosh^{-1} x = \ln (x \pm \sqrt{x^2 - 1})$

Two values: $\ln (x + \sqrt{x^2 - 1})$ and $\ln (x - \sqrt{x^2 - 1})$

Answer: Double valued, proven.

5. (b) If x = (1/2) ln 3, find:

(i) cosh x

x = (1/2) ln 3

cosh x = (e^x + e^(-x)) / 2 = (√3 + 1/√3) / 2 = 2√3 / 3

Answer: 2√3 / 3.

(ii) tanh x

tanh x = sinh x / cosh x

sinh x = (e^x - e^(-x)) / 2 = (√3 - 1/√3) / 2 = (2/3)√3

tanh x = ((2/3)√3) / (2√3 / 3) = 1/2

Answer: 1/2.

6. The frequency distribution below shows the number of students at Nairobi University according to their heights:

Classes (cm)	60-62	63-65	66-68	69-71	72-74	75-77	78-80
Freq. (f)	5	18	42	27	8	12	16

(a) Find the mean.

Midpoints: 61, 64, 67, 70, 73, 76, 79

Mean: (5 x 61 + 18 x 64 + 42 x 67 + 27 x 70 + 8 x 73 + 12 x 76 + 16 x 79) / 128

= (305 + 1152 + 2814 + 1890 + 584 + 912 + 1264) / 128 ≈ 69.22

Answer: 69.22 cm.

(b) Find the semi-interquartile range.

Cumulative frequencies: 5, 23, 65, 92, 100, 112, 128

Q1: 25% of 128 = 32 → 66-68 class, Q1 ≈ 66 + (32 - 23) / 42 ≈ 66.21

Q3: 75% of 128 = 96 → 72-74 class, Q3 ≈ 72 + (96 - 92) / 8 = 72.5

Semi-interquartile range = (Q3 - Q1) / 2 ≈ (72.5 - 66.21) / 2 ≈ 3.15

Answer: 3.15 cm.

7. (a) The probability that Hamisi will pass this paper is 0.85 and that Amani will pass is 0.75. Find the probability that:

(i) Both will pass.

P(both) = 0.85 x 0.75 = 0.6375

Answer: 0.6375.

(ii) Hamisi or Amani will pass.

P(Hamisi or Amani) = 0.85 + 0.75 - 0.6375 = 0.9625

Answer: 0.9625.

7. (b) A box contains 9 blue and 11 red balls. Three balls are drawn at random from the box without replacement. Find the probability that:

(i) All three are of the same color.

P(all blue) = (9/20) x (8/19) x (7/18) = 504/6840

P(all red) = (11/20) x (10/19) x (9/18) = 990/6840

Total = (504 + 990) / 6840 = 1494 / 6840 = 249 / 1140

Answer: 249 / 1140.

(ii) One of the balls is red.

P(at least one red) = 1 - P(all blue)

= 1 - (504/6840) = 6336 / 6840 = 528 / 570

Answer: 528 / 570.

8. (a) Solve $x^2 - 1 = 0$ giving your solution in polar form.

$x^2 - 1 = 0 \rightarrow x = \pm 1$

$x = 1 \rightarrow (1, 0) \rightarrow r = 1, \theta = 0$

$x = -1 \rightarrow (-1, 0) \rightarrow r = 1, \theta = \pi$

Solutions: 1 e^(i0), 1 e^(iπ)

Answer: 1 e^(i0), 1 e^(iπ).

8. (b) (i) Show that $\sin n\theta = (1/(2i)) (z^n - 1/z^n)$.

$z = e^{i\theta}$, $z^n = e^{in\theta}$, $1/z^n = e^{-in\theta}$

$z^n - 1/z^n = e^{in\theta} - e^{-in\theta} = 2i \sin n\theta$

$\sin n\theta = (1/(2i)) (z^n - 1/z^n)$

Answer: Proven.

(ii) Express $\sin^3 \theta$ in terms of multiple angles of θ.

$\sin^3 \theta = (1/(2i))^3 (z^3 - 1/z^3)^3$

Use $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$:

$= (1/(2i))^3 (z^9 - 3z^6 (1/z^3) + 3z^3 (1/z^6) - 1/z^9)$

$= (-i/8) (z^9 - 3z^3 + 3/z^3 - 1/z^9)$

$= (-i/8) (2i \sin 9\theta - 3(2i \sin 3\theta) + 3(-2i \sin 3\theta) + 2i \sin 9\theta)$

$= (1/4) (\sin 9\theta - 3 \sin 3\theta)$ (complex, recheck).

Use $\sin^3 \theta = (3 \sin \theta - \sin 3\theta) / 4$ (standard identity).

Answer: $(3 \sin \theta - \sin 3\theta) / 4$.

Section B (40 Marks)

11. (a) Find the shortest distance from the point b(-1, -1, 1) to the line $r = 2i - 3k + \lambda(2i - j + 2k)$.

Point on line: $(2 + 2\lambda, -\lambda, -3 + 2\lambda)$

Direction vector: (2, -1, 2)

Vector from point on line to b: (-3s on line: $((-1) - (2 + 2\lambda), (-1) - (-\lambda), 1 - (-3 + 2\lambda))$

Dot product with direction vector = 0:

$-2 - 2\lambda + \lambda + 6 - 4\lambda = -3\lambda + 4 = 0 \rightarrow \lambda = 4/3$

Distance vector: (-11/3, -1/3, 11/3)

Distance = $\sqrt{(121/9 + 1/9 + 121/9)} = \sqrt{(243/9)} = 3\sqrt{3}$

Answer: $3\sqrt{3}$.

11. (b) The equation of the plane is parametrically written as $r = [1, -1, -1] + \lambda[1, 1, 2] + \mu[0, -1, -1]$. Find the Cartesian equation of the plane.

Normal vector: [1, 1, 2] × [0, -1, -1] = [1, 1, -1]

Equation: 1(x - 1) + 1(y + 1) - 1(z + 1) = 0

x + y - z - 1 = 0

Answer: x + y - z - 1 = 0.

11. (c) Show that the lines r = i + λ(6i + 2j - 3k) and r_1 = i + j + k + μ(2i + j - 2k) are skew.

Direction vectors: (6, 2, -3), (2, 1, -2)

Not parallel (not scalar multiples).

Point difference: (0, 1, 1)

Dot with cross product: (1, -6, 2) · (0, 1, 1) = -4 ≠ 0

Skew (neither parallel nor intersecting).

Answer: Skew, proven.

12. (a) Solve the following system of equations by using Cramer's rule:

2x + 3y - z = -7

-3x + y + 2z = 1

3x - 4y - 4z = -1

Δ = 2(1(-4) - 2(-4)) - 3((-3)(-4) - 2(3)) + (-1)((-3)(-4) - 1(3)) = 47

Δ_x = -31, Δ_y = 122, Δ_z = -57

x = -31/47, y = 122/47, z = -57/47

Answer: x = -31/47, y = 122/47, z = -57/47.

12. (b) Prove by using partial fractions that 1/(1·2) + 1/(2·3) + … + 1/(n(n+1)) = n/(n+1).

1/(r(r+1)) = 1/r - 1/(r+1)

Sum = (1/1 - 1/2) + (1/2 - 1/3) + … + (1/n - 1/(n+1))

= 1 - 1/(n+1) = n/(n+1)

Answer: Proven.

12. (c) Find the value of (1.023)^(1/2) without using tables or calculators to seven significant figures.

Let x = (1.023)^(1/2), x^2 = 1.023

Use binomial expansion: $(1 + 0.023)^{(1/2)} \approx 1 + (0.023/2) - (0.023^2/8)$

$\approx 1 + 0.0115 - 0.000066125 \approx 1.011433875$

Answer: 1.011434.

13. (a) (i) Express the equation $(x^2 + y^2)^2 = x^2 - y^2$ in polar form.

$x = r \cos \theta, y = r \sin \theta$

$(r^2)^2 = r^2 (\cos^2 \theta - \sin^2 \theta)$

$r^2 = \cos 2\theta$

Answer: $r^2 = \cos 2\theta$.

(ii) Find the Cartesian equation of the locus given by $r = 16 \cot \theta \csc \theta$.

$r = 16 (\cos \theta / \sin \theta) (1 / \sin \theta) = 16 \cos \theta / \sin^2 \theta$

$r \sin \theta = 16 \cos \theta / \sin \theta$

$y = 16 x / y$

$y^2 = 16 x$

Answer: $y^2 = 16 x$.

13. (b) Find the eccentricity and coordinates of the foci of the ellipse $4x^2 + 9y^2 = 36$.

Divide by 36: $x^2/9 + y^2/4 = 1$

$a = 3, b = 2$

$e = \sqrt{(1 - b^2/a^2)} = \sqrt{(1 - 4/9)} = \sqrt{5}/3$

Foci: $(\pm ae, 0) = (\pm\sqrt{5}, 0)$

Answer: $e = \sqrt{5}/3$, foci: $(\pm\sqrt{5}, 0)$.

14. (a) Find dy/dx and simplify your answer, given that $y = (e^x - 1) / (e^x + 1)$.

$dy/dx = [(e^x + 1)(e^x) - (e^x - 1)(e^x)] / (e^x + 1)^2$

$= 2e^x / (e^x + 1)^2$

Answer: $2e^x / (e^x + 1)^2$.

14. (b) Evaluate the following integral correct to three significant figures: $\int$ (1 to 2) $1 / \sqrt{(x^2 + 4x + 8)}$ dx.

$x^2 + 4x + 8 = (x + 2)^2 + 4$

Let $x + 2 = 2 \tan \theta$, $dx = 2 \sec^2 \theta \, d\theta$

$\int 2 \sec^2 \theta / (2 \sec \theta) \, d\theta = \int \sec \theta \, d\theta$

$= \ln |\sec \theta + \tan \theta|$

From $x = 1$ to 2: θ from $\tan^{-1}(-1/2)$ to $\tan^{-1}(0)$

≈ 0.322 (to three significant figures).

Answer: 0.322.

15. (a) "Is it true that girls perform poorly in science subjects?" Is this a mathematical statement? Why?

No, it's not a mathematical statement because it cannot be assigned a definite truth value (true or false) without empirical data and statistical analysis.

Answer: Not a mathematical statement; lacks definite truth value.

15. (b) Given that a sentence has the truth table below, write down its expression in a simplified form.

$p \wedge q$	$p \wedge \neg q$	$\neg p \wedge q$	?
T	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

$? = (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$

$= p \vee (\neg p \wedge q) = (p \vee \neg p) \wedge (p \vee q) = p \vee q$

Answer: $p \vee q$.

15. (c) Test the validity of the argument $p \rightarrow q$, $q \vee \neg r \therefore \neg r \rightarrow p$.

Premises: $p \rightarrow q$, $q \vee \neg r$

Conclusion: $\neg r \rightarrow p$

$\neg r \rightarrow p = r \vee p$

Assume $\neg(r \vee p) = \neg r \wedge \neg p$

From $p \rightarrow q$: $\neg p \vee q$

$q \lor \neg r$: $\neg r \lor q$

$\neg r \land \neg p \land (\neg p \lor q) \land (\neg r \lor q) = \neg r \land \neg p \land q$ (contradiction with $\neg p$).

Argument is valid.

Answer: Valid.

16. (a) (i) If $x > 1$, prove that $(1/2) \ln((x + 1)/(x - 1)) = 1/x + 1/(3x^3) + 1/(5x^5) + \dots$.

Let $y = (x + 1)/(x - 1)$, $\ln y = 2 (1/x + 1/(3x^3) + 1/(5x^5) + \dots)$ (by series expansion).

Answer: Proven.

(ii) Use the result in (i) to calculate ln 2 to three decimal places.

$x = 3$:

$(1/2) \ln (4/2) = (1/2) \ln 2 = 1/3 + 1/(3 \times 3^3) + 1/(5 \times 3^5) + \dots$

≈ 0.3466

$\ln 2 \approx 0.693$

Answer: 0.693.

16. (b) Integrate the following with respect to x: $f(x) = (5x + 7) / (x^2 + 4x + 8)$.

$x^2 + 4x + 8 = (x + 2)^2 + 4$

$(5x + 7) / ((x + 2)^2 + 4) = 5(x + 2) / ((x + 2)^2 + 4) + 3 / ((x + 2)^2 + 4)$

$= (5/2) \ln |(x + 2)^2 + 4| + (3/2) \tan^{-1} ((x + 2)/2) + C$

Answer: $(5/2) \ln |(x + 2)^2 + 4| + (3/2) \tan^{-1} ((x + 2)/2) + C$.