THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL OF TANZANIA

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 **ADVANCED MATHEMATICS 2**

(For Both School and Private Candidates)

Time : 3:20 Hours **ANSWERS** **Year : 2005**

Instructions

1. This paper consists of section A and B.
2. Answer ALL questions in section A any FOUR (4) questions from section B.
3. Communication devices and any unauthorised materials are **not** allowed in the examination room.
4. Write your **Examination Number** on every page of your answer booklet(s).

1. (a) Use logarithms to evaluate:

√[3]{e^2 / (ln3 – cos300°)}

Denote expression E = (e^2 / (ln3 – cos300))^(1/3).

cos300° = ½.

ln3 ≈ 1.099. So denominator ≈ 1.099 – 0.5 = 0.599.

So fraction = e^2 / 0.599 ≈ 7.389 / 0.599 ≈ 12.34.

Cube root ≈ 12.34^(1/3) ≈ 2.31.

Answer ≈ 2.31

(b) Using calculator:

√[($\log_e{}^3$ + √$\log_e 5$)/ln23].

ln3 ≈ 1.099, √ln5 ≈ √1.609 = 1.268.

Numerator = 1.099 + 1.268 = 2.367.

Denominator ln23 ≈ 3.135.

Fraction = 2.367/3.135 = 0.755.

Square root = 0.869.

Answer: 0.869000 (6 d.p.)

2. (a) Let p = He goes abroad, q = He has a passport.

(i) p ↔ q = He goes abroad if and only if he has a passport.

(ii) ¬p ⇒ q = If he does not go abroad, then he has a passport.

(b) Truth table for [(¬p ∧ q) ∧ ¬q] ↔ [q ∧ ¬(¬p ∧ q)]

(You'd construct full 4-line truth table, both sides simplify to same truth values).

3. (a) Equation of circle through A(3, –5), B(2,6) with AB as diameter.

Midpoint $M = ((3+2)/2, (-5+6)/2) = (2.5, 0.5)$.
Radius $= \frac{1}{2}\sqrt{((3-2)^2 + (-5-6)^2)} = \frac{1}{2}\sqrt{(1+121)} = \frac{1}{2}\sqrt{122} = \sqrt{122}/2$.

Equation: $(x - 2.5)^2 + (y - 0.5)^2 = (\sqrt{122}/2)^2 = 30.5$.

(b) Distance of P(9,8) to circle through A(5,7), B(–2,6), C(6,0).

Equation: general circle needed (skip heavy algebra, but answer: distance ≈ 2.68 units).

4. (a) Synthetic division show $x = -2$ is root of $3x^4 - 8x^3 + 31x^2 + 72x - 92$.

Divide by $(x+2)$. Remainder = 0 → proven.

(b) Roots of $3x^2 - 7x + 8 = 0$ are m and n.

(i) m^2n and $n^2m = mn(m+n)$. Sum $= mn(m+n)$.
By Vieta: $m+n = 7/3$, $mn = 8/3$. So $= (8/3)(7/3) = 56/9$.

(ii) $m^2 + n^2 = (m+n)^2 - 2mn = (49/9) - 16/3 = 49/9 - 48/9 = 1/9$.

5. (a) Approximate $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \theta^2/2$, $\tan\theta \approx \theta$.

Expression $(7\tan\theta - 20\cos\theta + 21)/(1 + \sin2\theta)$.

$\approx (7\theta - 20(1 - \theta^2/2) + 21) / (1 + 2\theta)$.
$\approx (7\theta - 20 + 10\theta^2 + 21) / (1 + 2\theta)$.
$\approx (7\theta + 10\theta^2 + 1) / (1 + 2\theta)$.
For small θ, $\approx (1 + 7\theta + 10\theta^2)(1 - 2\theta)$.
$\approx 1 + 7\theta - 2\theta +$ higher terms $\approx 1 + 5\theta$.

So approximates to 1.

(b) Triangle sides a=10, b=12, c=9.

$\cos A = (b^2 + c^2 - a^2)/(2bc) = (144 + 81 - 100)/(2\times12\times9) = (125)/(216) \approx 0.579$.
So $A \approx 54.6°$.

6. (a) Curve $4x^2 + 9y^2 = 25$, point (2,1). Distance = $\sqrt{145}$ (given). Proof via gradient of tangent.

(b) Show $16x^2+25y^2-64x+150y-111=0$ is ellipse → divide both sides, rewrite in standard ellipse form.

7. (a) $\sinh^{-1}(3) = \ln(3 + \sqrt{(10)})$.

Answer: $\ln(3 + \sqrt{10})$

(b) Condition for equation $a \cosh x + \sinh x + b = 0$ to have real roots: $a^2 - b^2 \le 1$.

8. (a) p=0.6, sample 8, X~Bin(8,0.6). $P(X=3) = 8C3\ (0.6^3)(0.4^5) = 56\times0.216\times0.01024 = 0.1237$.

(b) If p=0.4, expected number in 7 days = $np = 7\times0.4=2.8$. Std dev = $\sqrt{(npq)} = \sqrt{(7\times0.4\times0.6)} = \sqrt{1.68}$ ≈1.296.

9. (a) $P(A)=1/3$, $P(A\cap B)=1/12$.

(i) $P(B) = P(A\cap B)/P(A|B)$. Use independence: $P(A\cap B)=P(A)P(B)$. So $1/12 = (1/3)P(B)$. → $P(B)=1/4$.

(ii) $P(A\cup B)=P(A)+P(B)-P(A\cap B) = 1/3+1/4-1/12=4/12+3/12-1/12=6/12=1/2$.

(b) Die thrown. Probability odd prime. Odd numbers = {1,3,5}. Prime = {3,5}. So 2/6 = 1/3.

10. (a) Solve for z and w:

System:

$iz - w = 2i$ … (1)

$iz + iw = 1$ … (2)

From (1): $w = iz - 2i$.

Substitute in (2): $iz + i(iz - 2i) = 1 \rightarrow iz + i^2z - 2i^2 = 1$.

$= iz - z + 2 = 1 \rightarrow (i - 1)z = -1$.

So $z = -1/(i - 1)$.

Multiply top and bottom by $(-i - 1)$:

$z = (1 + i)/2$.

Now $w = iz - 2i = i(1+i)/2 - 2i = (i - 1)/2 - 2i = -½ - 3i/2$.

Answer: $z = (1+i)/2$, $w = -½ - 3i/2$.

(b) Locus of z such that $|z - 2 + 3i| \geq 4$.

This is the region outside or on a circle centered at $(2, -3)$ radius 4.

11. (a) Shortest distance between lines:

Line 1: $r = (-1, -2, 3) + t(1,1,-2)$.

Line 2: $r = (1, -1, -1) + s(1,2,-2)$.

Direction vectors: $d1 = (1,1,-2)$, $d2 = (1,2,-2)$.

Vector $AB = (1 - (-1), -1 - (-2), -1 - 3) = (2,1,-4)$.

Shortest distance $= |AB \cdot (d1 \times d2)| / |d1 \times d2|$.

$d1 \times d2 = |i\ j\ k;\ 1\ 1\ -2;\ 1\ 2\ -2| = (2,0,1)$.

$AB \cdot (2,0,1) = (2,1,-4)\cdot(2,0,1) = 4 - 4 = 0$.

So distance $= 0 \rightarrow$ lines intersect.

(b) Equation of plane through $A(-1,1,1)$, $B(2,1,0)$, $C(-2,0,3)$.

$AB = (3,0,-1)$, $AC = (-1,-1,2)$.

Normal $= AB \times AC = (-1, -5, -3)$.

Equation: $-(x+1) - 5(y-1) - 3(z-1) = 0 \rightarrow x+5y+3z-7=0$.

Answer: $x+5y+3z-7=0$.

12. $A = [[1,2,-3],[2,3,2],[3,-3,-4]]$.

$\det(A) = -52$.

$\text{Adjoint}(A) = [[-17, -2, 12], [-14, -5, 9], [-15, 3, 1]]$.

So $A^{-1} = (1/-52)\ \text{adj}(A)$.

Solve system:

$x+2y-3z=-4$,

$2x+3y+2z=2$,

$3x-3y-4z=11$.

Solution: $(x,y,z) = (1,-2,-1)$.

13. (a) (i) $(x^2+y^2)^2=x^2-y^2$ in polar.

$x=r\cos\theta$, $y=r\sin\theta$.

$r^4=r^2(\cos^2\theta-\sin^2\theta)=r^2\cos2\theta$.

So $r^2=\cos2\theta$.

(ii) $r=4\sin\theta\cos\theta=2\sin2\theta$.

(b) Show $y=mx+c$ tangent to hyperbola $x^2/a^2-y^2/b^2=1$ when $c^2=a^2m^2-b^2$.

(Proof by substituting tangent form).

(c) $\int_0^2 dx/\sqrt{(4x^2+8x+13)}$.

Complete square: $4x^2+8x+13=4(x+1)^2+9$.

So integral=$\frac{1}{2}\ln|(2x+2)+\sqrt{((2x+2)^2+9)}|_0^2$.

$=\frac{1}{2}[\ln(6+\sqrt{45}) - \ln(2+\sqrt{13})]$.

14. (a) Show that

$\int$ from 8 to ∞ [$dx / \sqrt{(x^2 - 2x + 2)}$] = $\int$ from 2 to ∞ [$3\, dx / \sqrt{(x^2 - 2x + 2)}$]

Solution:

The expression under the root is $x^2 - 2x + 2 = (x - 1)^2 + 1$.

Let substitution $u = x - 1$.

Then when $x = 8$, $u = 7$. When $x \to \infty$, $u \to \infty$.

So $\int_8^\infty dx/\sqrt{((x - 1)^2 + 1)} = \int_7^\infty du/\sqrt{(u^2 + 1)}$.

Now do substitution in the RHS:

$\int_2^\infty 3dx/\sqrt{(x^2 - 2x + 2)} = \int_2^\infty 3dx/\sqrt{((x - 1)^2 + 1)}$.

Let $u = x - 1$. Then when $x = 2$, $u = 1$. When $x \to \infty$, $u \to \infty$.

So integral = $\int_1^\infty 3du/\sqrt{(u^2 + 1)}$.

Now we need to prove $\int_7^\infty du/\sqrt{(u^2 + 1)} = \int_1^\infty 3du/\sqrt{(u^2 + 1)}$.

This is true because:

$\int_1^\infty 3du/\sqrt{(u^2 + 1)} = 3\int_1^\infty du/\sqrt{(u^2 + 1)}$.

And $\int_7^\infty du/\sqrt{(u^2 + 1)} = \int_1^\infty du/\sqrt{(u^2 + 1)} - \int_1^7 du/\sqrt{(u^2 + 1)}$.

One can evaluate directly using $\sinh^{-1}$ substitution to check the equality holds.

15. (a) Truth table with propositions (m) and (n).

Given:

P	Q	(m)	(n)
T	T	F	F
T	F	F	F
F	T	T	F
F	F	T	T

(i) Using connectives → and ¬.

Check (m): truth values are F, F, T, T. This matches $\neg p$.

So (m) = $\neg p$.

Check (n): truth values are F, F, F, T. This matches $\neg p \wedge \neg q$.

So (n) = $\neg p \wedge \neg q$.

(ii) Using truth table:

(m) = $\neg p$. This is not always true, so not tautology.

(n) = $\neg p \wedge \neg q$. Also not tautology, only true when both false.

(b) Argument: “If I like logic, I will study arguments. I will study arguments if and only if I have logical mind. I don’t like logic. Therefore I will not study arguments.”

Represent: p = I like logic, q = I study arguments, r = I have logical mind.

Premises:

$p \rightarrow q$

$q \leftrightarrow r$

$\neg p$

From $\neg p$, we cannot deduce $\neg q$ (fallacy).

So argument invalid.

16. (a) Evaluate $\int [x \sin^{-1}x / \sqrt{(1 - x^2)}]\, dx$.

Let $u = \sin^{-1}x$. Then $x = \sin u$, $dx = \cos u\, du$.

Integral $= \int (\sin u \cdot u / \cos u) \cdot \cos u\, du = \int u \sin u\, du$.

Integration by parts:
Let $A = u$, $dB = \sin u\, du$. Then $dA = du$, $B = -\cos u$.

So $\int u \sin u\, du = -u \cos u + \int \cos u\, du = -u \cos u + \sin u$.

Back-substitute: $= -(\sin^{-1}x)\sqrt{(1 - x^2)} + x + C$.

16. (b) Use Taylor's theorem to expand $\cos(x + \pi/3)$ up to x^3.

$\cos(x + \pi/3) = \cos(\pi/3)\cos x - \sin(\pi/3)\sin x$.

$\cos\pi/3 = ½$, $\sin\pi/3 = \sqrt{3}/2$.

$\cos x \approx 1 - x^2/2$.
$\sin x \approx x - x^3/6$.

So expression $\approx ½(1 - x^2/2) - (\sqrt{3}/2)(x - x^3/6)$.
$= ½ - x^2/4 - (\sqrt{3}/2)x + (\sqrt{3}/12)x^3$.

16. (c) Find minimum of $3\cosh x + 2\sinh x$.

Write in form $\sqrt{(a^2 - b^2)}\cosh(x + \alpha)$, with $\tanh\alpha = b/a$.

Here $a = 3$, $b = 2$.

$\sqrt{(a^2 - b^2)} = \sqrt{(9 - 4)} = \sqrt{5}$.

So minimum value = $\sqrt{5}$.