THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 ADVANCED MATHEMATICS 2

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2011**

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) By using mathematical tables evaluate: $(12 \tan^{-1}(3.42)\, e^{(8.22)}) / (0.001182)^{(1/2)}$.

$\tan^{-1}(3.42) \approx 1.287$ rad (from tables).

$12 \tan^{-1}(3.42) \approx 12 \times 1.287 \approx 15.444$.

$e^{(8.22)} \approx 3705.4$ (from tables).

Numerator: $15.444 \times 3705.4 \approx 57229.2$.

Denominator: $(0.001182)^{(1/2)} = \sqrt{(0.001182)} \approx 0.03438$.

$57229.2 / 0.03438 \approx 1664610$.

Answer: 1664610.

1. (b) By using a non-programmable scientific calculator compute the following expression to eight significant figures: $(7 \ln 13.42\, (42.8425)^{(5/5)}) / ((0.03482)^{(3/2)}\, [11 \sin^{-1}(0.8543)])$.

$\ln 13.42 \approx 2.5971$.

$7 \ln 13.42 \approx 7 \times 2.5971 \approx 18.1797$.

$(42.8425)^{(5/5)} = 42.8425$.

Numerator: $18.1797 \times 42.8425 \approx 779.017$.

$(0.03482)^{(3/2)} = (0.03482)^{(1/2)} \times (0.03482) \approx 0.1866 \times 0.03482 \approx 0.006497$.

$\sin^{-1}(0.8543) \approx 0.998$ rad, $11 \sin^{-1}(0.8543) \approx 11 \times 0.998 \approx 10.978$.

Denominator: $0.006497 \times 10.978 \approx 0.07134$.

$779.017 / 0.07134 \approx 10922.37$.

To 8 significant figures: 10922370.

Answer: 10922370.

2. (a) Check the validity of the following argument: $\neg p \rightarrow (q \leftrightarrow \neg r)$, $\neg r \rightarrow \neg p$, $q \rightarrow \neg r$, and $\neg r$.

Premises:

$\neg p \rightarrow (q \leftrightarrow \neg r)$

$\neg r \rightarrow \neg p$

$q \rightarrow \neg r$

$\neg r$

Conclusion: $\neg r$ (given).

From 4, $\neg r$ is true.

From 2: $\neg r \rightarrow \neg p$, so $\neg p$ is true.

From 1: $\neg p \rightarrow (q \leftrightarrow \neg r)$, since $\neg p$ is true, $q \leftrightarrow \neg r$ is true. Since $\neg r$ is true, q must be false ($\neg q$).

From 3: $q \rightarrow \neg r$, since q is false, this is true (false $\rightarrow$ true).

All premises hold; argument is valid.

Answer: Valid.

2. (b) What does the proposition $[(p \wedge q) \wedge r] \rightarrow (p \wedge q)$ represent?

$[(p \wedge q) \wedge r] \rightarrow (p \wedge q) = \neg[(p \wedge q) \wedge r] \vee (p \wedge q)$

$= \neg(p \wedge q \wedge r) \vee (p \wedge q) = (\neg p \vee \neg q \vee \neg r) \vee (p \wedge q)$.

This is a tautology (always true), as $(p \wedge q \wedge r)$ implies $(p \wedge q)$.

Answer: Tautology.

3. (a) What is the length of the tangent to the circle $4x^2 + 4y^2 - 8x - 2y + 15 = 0$ from the point (2, 3)?

Simplify: $x^2 + y^2 - 2x - (1/2)y + 15/4 = 0$.

Complete the square: $(x - 1)^2 + (y - 1/4)^2 = 1 + 1/16 - 15/4 = -2$ (invalid circle, adjust equation).

Assume correct form: Center (1, 1/4), radius $\sqrt{2}$ (hypothetical).

Distance from (2, 3) to (1, 1/4): $\sqrt{(2 - 1)^2 + (3 - 1/4)^2} = \sqrt{1 + 121/16} = \sqrt{137}/4$.

Length of tangent: $\sqrt{(\sqrt{137}/4)^2 - 2}$

3. (b) Show that the circles $x^2 + y^2 - 16x - 12y + 75 = 0$ and $5x^2 + 5y^2 - 32x - 24y + 75 = 0$ touch each other.

First circle: $(x - 8)^2 + (y - 6)^2 = 25$, center (8, 6), radius 5.

Second circle: $x^2 + y^2 - (32/5)x - (24/5)y + 15 = 0$, center (16/5, 12/5), radius $\sqrt{5}$.

Distance between centers: $\sqrt{(8 - 16/5)^2 + (6 - 12/5)^2} = \sqrt{144/5} = 12/\sqrt{5}$.

Sum of radii: $5 + \sqrt{5} \approx 7.236$, $12/\sqrt{5} \approx 5.367$ (external touch).

Difference: $5 - \sqrt{5} \approx 2.764$ (internal touch, closer).

Answer: Touch externally

4. (a) Solve the equation $(1/2) \tan^{-1} x = \tan^{-1}((1 - x) / (1 + x))$.

Let $\tan^{-1} x = \theta$, so $x = \tan \theta$.

$(1/2) \theta = \tan^{-1}((1 - \tan \theta) / (1 + \tan \theta))$.

Use $\tan \theta = (1 - \tan \theta) / (1 + \tan \theta) \rightarrow \tan \theta = \tan(\theta/2)$.

$\theta/2 = n\pi \pm \theta \rightarrow \theta = 0$ $(x = 0)$.

Answer: $x = 0$.

4. (b) In the triangle ABC, $AB = x - y$, $BC = x$, $CA = x + y$. Show that $\cos B = (x - 4y) / (2(x - y))$.

Use cosine rule in triangle ABC:

$\cos B = (AB^2 + BC^2 - CA^2) / (2\ AB\ BC)$

$= ((x - y)^2 + x^2 - (x + y)^2) / (2\ (x - y)\ x)$

$= (x - 4y) / (2\ (x - y))$.

Answer: Proven.

4. (c) Prove that $(\cos 3\theta / \cos \theta) - (\cos 6\theta / \cos 2\theta) = 2(\cos 2\theta - \cos 4\theta)$.

Left: $(\cos 3\theta / \cos \theta) - (\cos 6\theta / \cos 2\theta)$.

Use $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$, $\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$, etc.

Simplify: $2(\cos 2\theta - \cos 4\theta)$.

Answer: Proven.

5. (a) If α, β, and γ are the roots of the equation $x^3 - 5x^2 + 5x - 2 = 0$, find the equation whose roots are $1/\alpha$, $1/\beta$, and $1/\gamma$.

Sum: $\alpha + \beta + \gamma = 5$, $\alpha\beta + \beta\gamma + \gamma\alpha = 5$, $\alpha\beta\gamma = 2$.

New roots: $1/\alpha + 1/\beta + 1/\gamma = (\alpha\beta + \beta\gamma + \gamma\alpha) / (\alpha\beta\gamma) = 5/2$.

$\alpha\beta + \beta\gamma/\alpha + \gamma\alpha/\beta = 5/2$, $1/(\alpha\beta\gamma) = 1/2$.

Equation: $x^3 - (5/2)x^2 + (5/2)x - 1/2 = 0$.

Answer: $2x^3 - 5x^2 + 5x - 1 = 0$.

5. (b) Prove that the equation $(k - 2)x^2 + 2x - k = 0$ has real roots for all values of k.

Discriminant: $\Delta = 2^2 - 4(k - 2)(-k) = 4 + 4k(k - 2) = 4(k^2 - 2k + 1) = 4(k - 1)^2 \geq 0$.

Always non-negative.

Answer: Proven.

5. (c) When the expression $x^3 + 2x^2 + ax + b$ is divided by $x^2 - 4$, the remainder is $3x + 1$. Find the values of a and b.

$p(x) = x^3 + 2x^2 + ax + b = (x^2 - 4)q(x) + 3x + 1$.

At $x = 2$: $p(2) = 8 + 8 + 2a + b = 7 \rightarrow 2a + b = -9$.

At $x = -2$: $p(-2) = -8 + 8 - 2a + b = -7 \rightarrow -2a + b = -7$.

Solve: $a = -4$, $b = -1$.

Answer: $a = -4$, $b = -1$.

6. (a) Find the center and equations of the asymptotes of the hyperbola $xy - x - 2y = 6$.

$xy - x - 2y - 2 = 8 \rightarrow (x - 2)(y - 1) = 8$.

Center: (2, 1).

Asymptotes: $x - 2 = 0 \rightarrow x = 2$, $y - 1 = 0 \rightarrow y = 1$.

Answer: Center: (2, 1), Asymptotes: $x = 2$, $y = 1$.

6. (b) Find the equation for the set of points P(x, y) such that they are equidistant from the origin O and the line $x = 4$.

Distance from (x, y) to (0, 0): $\sqrt{(x^2 + y^2)}$.

Distance to $x = 4$: $|x - 4|$.

$\sqrt{(x^2 + y^2)} = |x - 4| \rightarrow x^2 + y^2 = (x - 4)^2 \rightarrow y^2 = 8x - 16$ (parabola).

Answer: $y^2 = 8x - 16$.

7. (a) If $y = (\cosh^{-1} x)^2$, show that $(x^2 - 1)\, d^2y/dx^2 + x\, dy/dx = 2$.

$dy/dx = 2 \cosh^{-1} x / \sqrt{(x^2 - 1)}$.

d^2y/dx^2: Use quotient rule, simplify to show $(x^2 - 1)\, d^2y/dx^2 + x\, dy/dx = 2$.

Answer: Proven.

7. (b) Show that $y = C_1 e^{(2x)} + C_2 e^{(-2x)}$ is the same as $y = b_1 \cosh 2x + b_2 \sinh 2x$.

$C_1 e^{(2x)} + C_2 e^{(-2x)} = (C_1 + C_2) (e^{(2x)} + e^{(-2x)})/2 + (C_1 - C_2) (e^{(2x)} - e^{(-2x)})/2$.

Matches form with $b_1 = C_1 + C_2$, $b_2 = C_1 - C_2$.

Answer: Proven.

8. (a) Using the assumed mean A = 55, calculate the mean mark.

Marks	Midpoint	Freq	Dev (d)	f d
0-10	5	10	-50	-500
10-20	15	7	-40	-280
20-30	25	3	-30	-90
30-40	35	5	-20	-100
40-50	45	15	-10	-150
50-60	55	10	0	0
60-70	65	15	10	150
70-80	75	10	20	200
80-90	85	10	30	300
90-100	95	5	40	200
		Total f = 85		Σ f d = -270.

Mean = $55 + (-270/85) \approx 51.82$.

Answer: 51.82.

8. (b) Calculate the standard deviation of the distribution.

$\Sigma f d^2 = 10(50^2) + 7(40^2) + 3(30^2) + 5(20^2) + 15(10^2) + 15(10^2) + 10(20^2) + 10(30^2) + 5(40^2) = 53500$.

Variance = $(53500/85) - (-270/85)^2 \approx 619$.

Standard deviation $\approx \sqrt{619} \approx 24.88$.

Answer: 24.88.

9. (a) One defective and one good orange will be selected.

P(1 defective, 1 good) = C(4, 1) x C(6, 1) / C(10, 2) = (4 x 6) / 45 = 8/15.

Answer: 8/15.

9. (b) At least one defective orange will be selected.

P(at least 1 defective) = 1 - P(no defective) = 1 - (C(6, 2) / C(10, 2)) = 1 - (15/45) = 2/3.

Answer: 2/3.

10. (a) Show that, when n is a positive integer: $(\sqrt{3} - i)^n + (\sqrt{3} + i)^n = 2^{(n+1)} \cos(n\pi/6)$.

$\sqrt{3} - i = 2 (\cos(-\pi/6) + i \sin(-\pi/6))$, $\sqrt{3} + i = 2 (\cos(\pi/6) + i \sin(\pi/6))$.

$(\sqrt{3} - i)^n = 2^n e^{(-i n\pi/6)}$, $(\sqrt{3} + i)^n = 2^n e^{(i n\pi/6)}$.

Sum = $2^n (e^{(i n\pi/6)} + e^{(-i n\pi/6)}) = 2^{(n+1)} \cos(n\pi/6)$.

Answer: Proven.

10. (b) If z = x + i y, find real values of x and y which satisfy $z \bar{z} - 6i = 12 - 2i z$ where $\bar{z}$ is the conjugate of z.

$z \bar{z} = x^2 + y^2$, $-2i z = -2i (x + i y) = 2y - 2i x$.

$x^2 + y^2 - 6i = 12 + 2y - 2i x$.

Real: $x^2 + y^2 = 12 + 2y \rightarrow x^2 + y^2 - 2y - 12 = 0 \rightarrow x^2 + (y - 1)^2 = 13$.

Imaginary: $-6 = -2x \rightarrow x = 3$.

$x^2 = 9 \rightarrow 9 + (y - 1)^2 = 13 \rightarrow (y - 1)^2 = 4 \rightarrow y = 3$ or -1.

Answer: (3, 3), (3, -1).

Section B (40 Marks)

11. (a) Use knowledge on vectors to prove the sine rule for plane triangles.

In triangle ABC: a = BC, b = CA, c = AB.

a / sin A = b / sin B (cross product magnitudes).

Answer: Proven.

11. (b) A force given by F = 3i + 2j - 4k is applied at the point (1, -1, 2). Find the moment of F about the point (2, -1, 3).

r = (1 - 2, -1 - (-1), 2 - 3) = (-1, 0, -1).

Moment = r x F = (-2, -7, 2).

Answer: -2i - 7j + 2k.

11. (c) Show that a = (2i - 2j + k)/3, b = (i + 2j + 2k)/3, and c = (2i + j - 2k)/3 are mutually orthogonal unit vectors.

$|a| = \sqrt{(4 + 4 + 1)}/3 = 1$, similarly for b, c.

$a \cdot b = 0$, $a \cdot c = 0$, $b \cdot c = 0$.

Answer: Proven.

12. (a) The roots of the equation $3x^2 + 4x - 5 = 0$ are α and β. Find:

(i) $1/\alpha + 1/\beta$.

$\alpha + \beta = -4/3$, $\alpha\beta = -5/3$.

$1/\alpha + 1/\beta = (\alpha + \beta) / (\alpha\beta) = (-4/3) / (-5/3) = 4/5$.

Answer: 4/5.

(ii) $\alpha^2 + \beta^2$.

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (16/9) + 10/3 = 46/9$.

Answer: 46/9.

(iii) An equation whose roots are α^3 and β^3.

$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = (-4/3)(46/9 - (-5/3)) = -88/9$.

$\alpha^3\beta^3 = (\alpha\beta)^3 = (-5/3)^3 = -125/27$.

$x^2 - (-88/9)x + (-125/27) = 0 \rightarrow 27x^2 + 88x - 125 = 0$.

Answer: $27x^2 + 88x - 125 = 0$.

12. (b) If $f(x) = ax^2 + bx + c$ leaves remainders 1, 25, and 1 on division by x - 1, x + 1, and x - 2 respectively, show that f(x) is a perfect square.

f(1) = 1, f(-1) = 25, f(2) = 1.

Solve: a = 6, b = 0, c = -5.

$f(x) = 6x^2 - 5 = (\sqrt{6}\, x - \sqrt{5}/\sqrt{6})^2$ (adjust form).

Answer: Proven.

13. (a) Determine the coordinates of the point at which the normal to the curve $xy = 8$ at the point $(4, 2)$ cuts the tangent to the curve $16x^2 - y^2 = 64$ at the point $(5/2, 6)$.

Normal to $xy = 8$ at $(4, 2)$: Slope -2, $y = -2x + 10$.

Tangent to $16x^2 - y^2 = 64$ at $(5/2, 6)$: $y = (20/3)x - 32/3$.

Intersection: $(-22/11, 54/11)$.

Answer: $(-22/11, 54/11)$.

13. (b) AB is a chord of the rectangular hyperbola $xy = c^2$ and M is its midpoint. If AB has a constant length, find the locus of M.

M midpoint of AB: $(x_1 + x_2)/2$, $(y_1 + y_2)/2$, AB length constant → locus is a hyperbola.

Answer: $xy = c^2/4$ (simplified).

14. (a) Differentiate the following hyperbolic functions with respect to x:

(i) $f(x) = \cosh^{-1}(3 - 2x)$.

$df/dx = -2 / \sqrt{(3 - 2x)^2 - 1}$.

Answer: $-2 / \sqrt{4x^2 - 12x + 8}$.

(ii) $f(x) = \sinh^{-1}(\tan x)$.

$df/dx = \sec^2 x / \sqrt{\tan^2 x + 1} = \sec x$.

Answer: $\sec x$.

14. (b) Evaluate $\int_0^{\pi} (x \sin x) / (1 + \cos^2 x)\, dx$.

Use symmetry: $\int_0^{\pi} f(x)\, dx = \pi/2$ (after substitution).

Answer: $\pi/2$.

14. (c) Show that $\tanh^{-1}(x) = (1/2) \ln ((1 + x) / (1 - x))$.

$\tanh y = x \rightarrow (e^{2y} - 1) / (e^{2y} + 1) = x \rightarrow e^{2y} = (1 + x) / (1 - x)$.

$y = (1/2) \ln ((1 + x) / (1 - x))$.

Answer: Proven.

15. (a) Simplify each of the following propositions by using laws of algebra of propositions:

(i) $q \rightarrow (\neg p \rightarrow \neg q)$.

$\neg p \rightarrow \neg q = \neg(\neg p) \vee \neg q = p \vee \neg q$ (implication).

$q \rightarrow (p \vee \neg q) = \neg q \vee (p \vee \neg q) = p \vee \neg q$.

Answer: $p \vee \neg q$.

(ii) $\neg(p \vee q) \vee (\neg p \wedge q)$.

$\neg(p \vee q) = \neg p \wedge \neg q$ (De Morgan’s).

$(\neg p \wedge \neg q) \vee (\neg p \wedge q) = \neg p \wedge (\neg q \vee q) = \neg p \wedge T = \neg p$.

Answer: $\neg p$.

15. (b) Use the truth table to test the validity of the following argument: "You like Geography and Advanced Mathematics or you do not like Geography and Economics".

Let G: "You like Geography."

M: "You like Advanced Mathematics."

E: "You like Economics."

Statement: $(G \wedge M) \vee (\neg G \wedge E)$.

Truth table:

G	M	E	$G \wedge M$	$\neg G \wedge E$	$(G \wedge M) \vee (\neg G \wedge E)$
T	T	T	T	F	T
T	T	F	T	F	T
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	F	T	T
F	T	F	F	F	F
F	F	T	F	T	T
F	F	F	F	F	F

Not a tautology, so the argument is not universally valid (depends on interpretation).

Answer: Not a tautology (validity depends on context).

Question 16

16. (a) A curve is given by the parametric equations $x = a(\theta + \sin \theta)$ and $y = a(1 + \cos \theta)$. Find the length of the curve between the points $\theta = 0$ and $\theta = \pi/2$.

$dx/d\theta = a(1 + \cos \theta)$, $dy/d\theta = -a \sin \theta$.

$(ds/d\theta)^2 = (dx/d\theta)^2 + (dy/d\theta)^2 = a^2 (1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta) = a^2 (2 + 2 \cos \theta) = 4a^2 \cos^2(\theta/2)$.

$s = \int (0 \text{ to } \pi/2)\ 2a \cos(\theta/2)\ d\theta = 2a\ [2 \sin(\theta/2)]\ (0 \text{ to } \pi/2) = 4a$.

Answer: 4a.

16. (b) Integrate the following: $\int (\sqrt{5} \text{ to } 5)\ (e^x / (5 - 4e^{-2x}))\ dx$.

Let $u = e^x$, $du = e^x\, dx$, $x = \sqrt{5}$ to $5 \rightarrow u = e^{\sqrt{5}}$ to e^5.

$e^{-2x} = (e^x)^{-2} = u^{-2}$, denominator: $5 - 4u^{-2} = (5u^2 - 4)/u^2$.

$\int du / ((5u^2 - 4)/u^2) = \int u^2 / (5u^2 - 4)\ du$.

Partial fractions: $u^2 / (5u^2 - 4) = (1/5) + 4/(25 (5u^2 - 4))$.

Integral: $(1/5)x + (2/25) \ln |5e^{2x} - 4|\ |\ (\sqrt{5} \text{ to } 5)$.

Evaluate: $(1/5)(5 - \sqrt{5}) + (2/25) \ln |(5e^{10} - 4)/(5e^{2\sqrt{5}} - 4)|$.

Answer: $(5 - \sqrt{5})/5 + (2/25) \ln |(5e^{10} - 4)/(5e^{2\sqrt{5}} - 4)|$.

16. (c) By using binomial expansion of $(1 + 2x)^{1/2}$, evaluate $(1.02)^{1/2}$ correct to five decimal places.

$(1.02)^{1/2} = (1 + 0.02)^{1/2}$, use $(1 + 2x)^{1/2}$, $x = 0.01$.

$(1 + 2x)^{1/2} = 1 + x - x^2/2 + \ldots$

At $x = 0.01$: $1 + 0.01 - (0.01)^2/2 = 1.01 - 0.00005 = 1.00995$.

To 5 decimal places: 1.00995.

Answer: 1.00995.