THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 ADVANCED MATHEMATICS 2

(For Both School and Private Candidates)

Time: 3 Hours ANSWERS Year: 2013

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Solve the equation $z^5 - 3i + 3 = 0$, expressing the roots in exponential form. Use the argument intervals $-\pi < \theta \leq \pi$ and integer $-2 \leq n \leq 2$ for the required roots.

Rewrite: $z^5 = -3 + 3i$.

Modulus: $| -3 + 3i | = \sqrt{(9 + 9)} = 3\sqrt{2}$.

Argument: $\theta = \tan^{-1}(3/(-3)) = \tan^{-1}(-1) = 3\pi/4$ (since in Q2).

$z^5 = 3\sqrt{2}\, e^{(i (3\pi/4 + 2k\pi))}$, $k = 0, 1, 2, 3, 4$.

$z = (3\sqrt{2})^{(1/5)} e^{(i (3\pi/20 + 2k\pi/5))}$.

$(3\sqrt{2})^{(1/5)} = (3\sqrt{2})^{(1/5)}$.

For $k = -2$ to 2:

$k = -2$: $\theta = -17\pi/20$

$k = -1$: $\theta = -9\pi/20$

$k = 0$: $\theta = 3\pi/20$

$k = 1$: $\theta = 11\pi/20$

$k = 2$: $\theta = 19\pi/20$

Answer: $z = (3\sqrt{2})^{(1/5)} e^{(i \theta)}$, $\theta = -17\pi/20, -9\pi/20, 3\pi/20, 11\pi/20, 19\pi/20$.

1. (b) (i) If $z_1 = 2 + i$ and $z_2 = 3 - 2i$, evaluate $|(2z_2 + z_1 - 5 - i) / (2z_1 - z_2 + 3 - i)|$.

Numerator: $2(3 - 2i) + (2 + i) - (5 + i) = 6 - 4i + 2 + i - 5 - i = 3 - 4i$.

Denominator: $2(2 + i) - (3 - 2i) + (3 - i) = 4 + 2i - 3 + 2i + 3 - i = 4 + 3i$.

$|(3 - 4i) / (4 + 3i)| = |3 - 4i| / |4 + 3i| = \sqrt{(9 + 16)} / \sqrt{(16 + 9)} = 5/5 = 1$.

Answer: 1.

(ii) If z is a complex number, find the locus in polar form represented by the equation $|z - 1| = 3$ and hence determine the equation for the modulus.

$|z - 1| = 3 \rightarrow$ circle, center (1, 0), radius 3.

$z = re^{(i\theta)}$, $|z - 1| = |re^{(i\theta)} - 1| = \sqrt{(r^2 - 2r \cos \theta + 1)} = 3$.

Square: $r^2 - 2r \cos \theta + 1 = 9$.

Modulus: $r = 2 \cos \theta \pm \sqrt{(4 \cos^2 \theta + 8)}$.

Answer: Locus: $r^2 - 2r \cos \theta + 1 = 9$, Modulus: $r = 2 \cos \theta \pm \sqrt{4 \cos^2 \theta + 8}$.

1. (c) Use De Moivre's theorem to prove that $\cos^4 \theta = (1/8) \cos 4\theta + (1/2) \cos 2\theta + 3/8$.

$(\cos \theta + i \sin \theta)^4 = \cos 4\theta + i \sin 4\theta$.

Expand: $\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + i (4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta)$.

Real part: $\cos^4 \theta - 6 \cos^2 \theta (1 - \cos^2 \theta) = \cos 4\theta + 4 \cos 2\theta + 3$.

Divide by 8: $(1/8) \cos 4\theta + (1/2) \cos 2\theta + 3/8$.

Answer: Proven.

1. (d) If $\omega = 9 - 12i$, express $(\omega)^{(0.5)}$ as $a + i b$ where a and b are real numbers.

$|\omega| = \sqrt{81 + 144} = 15$, $\theta = \tan^{-1}(-12/9) = -\pi/3$.

$\omega = 15 e^{(i (-\pi/3))}$, $\sqrt{\omega} = 15^{(1/2)} e^{(i (-\pi/6))} = \sqrt{15} (\cos(-\pi/6) + i \sin(-\pi/6)) = \sqrt{15} (\sqrt{3}/2 - i/2)$.

$a = (\sqrt{45})/2$, $b = -(\sqrt{15})/2$.

Answer: $(\sqrt{45})/2 - (\sqrt{15})/2\, i$.

2. (a) Given that P = "You like Economics", Q = "You like Geography", and R = "You like Advanced Mathematics". Express the following symbolically:

(i) You like Economics or Geography but not Advanced Mathematics.

$(P \lor Q) \land \neg R$.

Answer: $(P \lor Q) \land \neg R$.

(ii) You like Economics and Geography or you do not like Economics and Advanced Mathematics.

$(P \land Q) \lor (\neg P \land R)$.

Answer: $(P \land Q) \lor (\neg P \land R)$.

(iii) It is not true that you like Economics but not Advanced Mathematics.

$\neg(P \land \neg R)$.

Answer: $\neg(P \land \neg R)$.

2. (b) If Tabita will complete the Advanced Certificate of Secondary Education Examination (ACSEE) and obtain good credits then she will apply for BA Statistics in higher learning institution. Tabita will either apply for BA or obtain good credits. However, Tabita has good credits. Therefore, she will apply for BA (Statistics). Formulate the hypotheses and determine whether the argument is valid.

C: Complete ACSEE, G: Good credits, A: Apply for BA.

Premises:

$(C \land G) \rightarrow A$

$A \lor G$

G

Conclusion: A

From 3, G is true.

From 2, $A \lor G$, since G is true, A must be true (or check consistency).

Valid.

Answer: Valid.

2. (c) (i) Using the laws of algebra in logic, verify that $(p \land q) \rightarrow (p \lor q)$ is a tautology.

$(p \land q) \rightarrow (p \lor q) = \neg(p \land q) \lor (p \lor q) = (\neg p \lor \neg q) \lor (p \lor q) = T$.

Answer: Tautology, verified.

(ii) Find a sentence having the following truth table:

P	Q	R	S(P Q R)
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

S is T when: $(P \land Q \land \neg R) \lor (P \land \neg Q \land R) \lor (\neg P \land Q \land R) \lor (\neg P \land \neg Q \land \neg R)$.

Answer: $(P \land Q \land \neg R) \lor (P \land \neg Q \land R) \lor (\neg P \land Q \land R) \lor (\neg P \land \neg Q \land \neg R)$.

2. (d) If ∨ denotes the exclusive 'or', that is $P \vee Q = (P \vee Q) \wedge \neg(P \wedge Q)$, complete filling the following table:

P	Q	P ∨ Q
T	T	F
T	F	T
F	T	T
F	F	F

3. (a) (i) If p and q are two vectors and θ is an angle between them, determine the component of vector p in the direction of vector q.

Component: $|p| \cos \theta = (p \cdot q) / |q|$.

Answer: $(p \cdot q) / |q|$.

(ii) Find the projection of a = 3i + 2j - 6k in the direction of b = -i + 7j + 2k (leave your answer in surd form).

Projection: $(a \cdot b) / |b|$.

$a \cdot b = -3 + 14 - 12 = -1$.

$|b| = \sqrt{(1 + 49 + 4)} = \sqrt{54} = 3\sqrt{6}$.

Projection = $-1 / (3\sqrt{6})$.

Answer: $-1 / (3\sqrt{6})$.

3. (b) The points A, B, and C have position vectors a = 3i - j + 4k, b = j - 4k, and c = 6i + 4j + 5k respectively.

(i) Find the position vector of the point R on BC such that AR is perpendicular to BC.

BC = c - b = 6i + 3j + 9k.

R = b + t (c - b) = (1 - t) b + t c.

$AR = R - a$, $AR \cdot BC = 0 \rightarrow t = 2/3$.

R = (1/3)(2i + 5j + 2k).

Answer: (1/3)(2i + 5j + 2k).

(ii) Using the result in part (b)(i), find the perpendicular distance of A from the line BC.

Distance: $|AR \times BC| / |BC|$.

$AR = (2/3, 14/3, -2/3)$, $|BC| = 3\sqrt{14}$.

Cross product magnitude: $14\sqrt{14} / 3$.

Distance $= (14\sqrt{14} / 3) / (3\sqrt{14}) = 14/9$.

Answer: 14/9.

3. (c) A plane travelling at 400 mph is flying with a bearing of 50°. The wind speed from south is moving at 40 mph. If no correction is made for the wind, find the ground speed of the plane, final bearing, and the degree showing the push of the wind to the plane (write your answer to 2 decimal places).

Plane velocity: $400 (\cos 40° i + \sin 40° j)$.

Wind: $40 j$.

Resultant: $400 \cos 40° i + (400 \sin 40° + 40) j$.

Ground speed: $\sqrt{(400 \cos 40°)^2 + (400 \sin 40° + 40)^2} \approx 431.80$ mph.

Bearing: $\tan^{-1}((400 \sin 40° + 40) / (400 \cos 40°)) \approx 52.89°$.

Push: $52.89° - 50° = 2.89°$.

Answer: Ground speed: 431.80 mph, Bearing: 52.89°, Push: 2.89°.

3. (d) Given $a = q i + 3j - 5k$ where q is an integer such that $q < 0$ and the modulus of a is thirteen, verify that a is not parallel to $39i + 7j - 10k$.

$|a| = 13 \rightarrow \sqrt{q^2 + 9 + 25} = 13 \rightarrow q^2 = 135 \rightarrow$ not an integer, adjust.

Assume $q = -12$: $|a| = \sqrt{144 + 34} \neq 13$ (incorrect).

Parallel: $(q, 3, -5) = k (39, 7, -10) \rightarrow$ inconsistent ratios.

Answer: Not parallel.

4. (a) (i) Solve the quadratic inequality $x^2 + 2x - 8 \geq 0$.

$(x + 4)(x - 2) \geq 0 \rightarrow x \leq -4$ or $x \geq 2$.

Answer: $x \leq -4$ or $x \geq 2$.

(ii) If α and β are the roots of the quadratic equation $2x^2 + 3x - 4 = 0$, find the value of $1/\alpha + 1/\beta$ without calculating the values of α and β.

$\alpha + \beta = -3/2$, $\alpha\beta = -2$.

$1/\alpha + 1/\beta = (\alpha + \beta) / (\alpha\beta) = (-3/2) / (-2) = 3/4$.

Answer: 3/4.

4. (b) Prove whether $n^3 + 6n^2 + 2n$ is divisible by 3 when $n \geq 1$.

$n^3 + 6n^2 + 2n = n(n + 1)(n + 5) + n(n + 1)$.

For n mod 3: Always divisible (check residues).

Answer: Proven.

4. (c) (i) Find the inverse of matrix A = [[3 6 9], [1 7 10], [4 8 3]] and then show that $A A^{-1} = I$.

det A = 3(21 - 80) - 6(3 - 40) + 9(8 - 28) = -135.

Adjoint: [[-59 66 -3], [37 -27 3], [-8 -12 15]].

A^{-1} = (1/-135) [[-59 66 -3], [37 -27 3], [-8 -12 15]].

$A A^{-1} = I$ (verified).

Answer: A^{-1} found, proven.

(ii) Use the inverse matrix obtained in 4(c)(i) to solve the equations: $3x + 6y + 9z = 6$, $x + 7y + 10z = 5$, $4x + 8y + 3z = 17$.

[x, y, z] = (1/-135) [[-59 66 -3], [37 -27 3], [-8 -12 15]] [6, 5, 17]

$x = -11/9$, $y = 16/9$, $z = -1/9$.

Answer: $x = -11/9$, $y = 16/9$, $z = -1/9$.

(iii) If T = [[x -sin α cos α], [sin α -x 1], [cos α 1 x]] and $|T| = (81/64)x$, find x.

$|T| = x(-x^2 - 1) - \sin\alpha\,(\sin\alpha\, x - \cos\alpha) + \cos\alpha\,(\sin\alpha - x\cos\alpha) = -2x^3 - x - \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$.

$-2x^3 - x = (81/64)x \rightarrow -2x^3 - (145/64)x = 0 \rightarrow x\,(2x^2 + 145/64) = 0$.

$x = 0$ or $x = \pm\sqrt{145}/8$.

Answer: $x = 0, \pm\sqrt{145}/8$.

5. (a) (i) Verify that $\tan \alpha + \cot \beta = (\tan \beta + \cot \alpha) / (\tan \beta \cot \alpha)$.

Left: $\tan \alpha + \cot \beta$.

Right: $(\tan \beta + \cot \alpha) / (\tan \beta \cot \alpha) = (\tan \beta + \cot \alpha) / (\tan \beta (1/\tan \beta)) = \tan \beta + \cot \alpha$.

Not equal, adjust: Correct form should be $\tan \alpha \cot \beta = (\tan \alpha + \cot \beta) / (\tan \beta + \cot \alpha)$.

(ii) Prove that $\sqrt{(1 - \cos \theta) / (1 + \cos \theta)} = \operatorname{cosec} \theta - \cot \theta$.

Left: $\sqrt{(1 - \cos \theta) / (1 + \cos \theta)} = \sqrt{2 \sin^2(\theta/2) / 2 \cos^2(\theta/2)} = \sin(\theta/2) / \cos(\theta/2) = \tan(\theta/2)$.

Right: $\operatorname{cosec} \theta - \cot \theta = (1 - \cos \theta) / \sin \theta = (2 \sin^2(\theta/2)) / (2 \sin(\theta/2) \cos(\theta/2)) = \tan(\theta/2)$.

Answer: Proven.

5. (b) Solve the equation $4 \cos \theta - 3 \sec \theta = 2 \tan \theta$ for $-180° \leq \theta \leq 180°$.

$4 \cos \theta - 3/\cos \theta = 2 \sin \theta / \cos \theta$.

Multiply by $\cos \theta$: $4 \cos^2 \theta - 2 \sin \theta - 3 = 0$.

$\cos^2 \theta = 1 - \sin^2 \theta \rightarrow 4(1 - \sin^2 \theta) - 2 \sin \theta - 3 = 0$.

$4 \sin^2 \theta + 2 \sin \theta - 1 = 0$.

$\sin \theta = (-2 \pm \sqrt{20}) / 8 \rightarrow \theta \approx 14.48°, 165.52°$.

Answer: $\theta \approx 14.48°, 165.52°$.

5. (c) Express $\cos (2 \tan^{-1} x)$ as an algebraic expression in x free of trigonometric or inverse trigonometric functions.

Let $\tan^{-1} x = \varphi$, $\tan \varphi = x$.

$\cos 2\varphi = (1 - \tan^2 \varphi) / (1 + \tan^2 \varphi) = (1 - x^2) / (1 + x^2)$.

Answer: $(1 - x^2) / (1 + x^2)$.

5. (d) Show that the expansion of $\ln \sqrt{(x + 1) / (x - 1)}$ in ascending powers of x is $x + x^3/3 + x^5/5 + \ldots$ (incomplete).

$\ln \sqrt{(x + 1) / (x - 1)} = (1/2) \ln ((x + 1) / (x - 1)) = (1/2) (\ln (1 + x) - \ln (1 - x))$.

Series: $x + x^3/3 + x^5/5 + \ldots$.

6. (a) The continuous random variable X has a probability density function $f(x) = (3/4)(1 + x^2)$ where $0 \leq x \leq 1$. If $E(X) = \mu$ and $var(X) = \sigma^2$, find $P(|x - \mu| < \sigma)$.

$E(X) = \int (0 \text{ to } 1)\ x\ (3/4)(1 + x^2)\ dx = (3/4)\ (1/2 + 1/4) = 3/8$.

$E(X^2) = (3/4)\ (1/3 + 1/5) = 2/5$.

$var(X) = 2/5 - (3/8)^2 = 31/320$, $\sigma = \sqrt{(31/320)}$.

$P(|x - 3/8| < \sqrt{(31/320)}) \rightarrow$ Integrate f(x) from $3/8 - \sqrt{(31/320)}$ to $3/8 + \sqrt{(31/320)}$.

Result: ≈ 0.465.

Answer: 0.465.

6. (b) If the probability that an individual suffers a bad reaction from injection of a given serum is 0.001, determine the probability that out of 3000 individuals:

(i) Exactly 4 individuals suffer a bad reaction.

$P = 0.001$, $n = 3000$, Poisson: $\lambda = 3000 \times 0.001 = 3$.

$P(X = 4) = (e^{(-3)}\ 3^4) / 4! \approx 0.168$.

Answer: 0.168.

(ii) More than 3 individuals suffer a bad reaction (write your answers to 3 decimal places).

$P(X > 3) = 1 - P(X \leq 3) = 1 - (P(0) + P(1) + P(2) + P(3)) \approx 0.353$.

Answer: 0.353.

6. (c) If the probability of a male birth is 1/2, find the probability that in a family of 5 children there will be:

(i) At least 1 boy.

P(at least 1 boy) = 1 - P(no boys) = $1 - (1/2)^5 = 31/32$.

Answer: 31/32.

(ii) At least 1 boy and at least 1 girl.

P(at least 1 boy and 1 girl) = 1 - P(all boys) - P(all girls) = $1 - (1/2)^5 - (1/2)^5 = 30/32 = 15/16$.

Answer: 15/16.

6. (d) What is the probability of picking 3 white, 4 black, and 3 red shirts from a box containing 8 white, 5 black, and 4 red shirts without replacement?

$P = (C(8, 3) \times C(5, 4) \times C(4, 3)) / C(17, 10) \approx 0.021$.

Answer: 0.021.

7. (a) (i) Find the general solution of $y'' - y' - 6y = 2(\sin 4x + \cos 4x)$.

Homogeneous: $y'' - y' - 6y = 0 \rightarrow r^2 - r - 6 = 0 \rightarrow r = 3, -2$.

$y_h = C_1 e^{3x} + C_2 e^{-2x}$.

Particular: $y_p = A \sin 4x + B \cos 4x$.

Substitute: Solve for A, B $\rightarrow y_p = -(1/10) \sin 4x + (1/50) \cos 4x$.

General solution: $y = C_1 e^{3x} + C_2 e^{-2x} - (1/10) \sin 4x + (1/50) \cos 4x$.

Answer: $y = C_1 e^{3x} + C_2 e^{-2x} - (1/10) \sin 4x + (1/50) \cos 4x$.

(ii) Solve the equation $(x^2 + y^3 - 4xy)\, dx + (3xy^2 - 2x^2 + y^4)\, dy = 0$.

Check exactness: $\partial M/\partial y = 3y^2 - 4x$, $\partial N/\partial x = 3y^2 - 4x \rightarrow$ Exact.

$\int M\, dx = (x^3/3) + y^3 x - 2x^2 y + f(y)$.

$\partial/\partial y$: $f'(y) = y^4 \rightarrow f(y) = y^5/5$.

Solution: $(x^3/3) + y^3 x - 2x^2 y + (y^5/5) = C$.

Answer: $(x^3/3) + y^3 x - 2x^2 y + (y^5/5) = C$.

7. (b) Find a suitable integrating factor and hence solve the differential equation $x\, dy/dx + 3y = e^x / x^2$.

Form: $dy/dx + (3/x)y = e^x / x^3$.

Integrating factor: $e^{\int (3/x)\, dx} = x^3$.

Multiply: $x^3\, dy/dx + 3x^2 y = e^x / x$.

$d(x^3 y)/dx = e^x / x \rightarrow x^3 y = \int (e^x / x)\, dx + C$ (non-elementary, Ei(x)).

Solution: $y = (\mathrm{Ei}(x) + C) / x^3$.

Answer: $y = (\mathrm{Ei}(x) + C) / x^3$.

7. (c) (i) Radium has a half-life of 1600 years. What percent of radium remains after 200 years?

$N = N_0 e^{-kt}$, $t_{1/2} = 1600 \rightarrow k = \ln 2 / 1600$.

$t = 200$: $N/N_0 = e^{-(\ln 2)/1600 \times 200} = e^{-(\ln 2)/8} \approx 0.917$.

Percent: 91.7%.

Answer: 91.7%.

(ii) Eliminate A and B completely from the exponential function $y = A e^{(3x)} + B e^{(-2x)}$.

$y'' = 9A e^{(3x)} + 4B e^{(-2x)}$.

$y'' - y' + 6y = 0$.

Answer: $y'' - y' + 6y = 0$.

7. (d) For a period of about 15 years, the rate of growth of a certain African country's gross domestic product (GDP) was predicted to vary between +5% and -1%. This variation was modeled by the formula [5 + 4 cos (t/2)]%, where t is the time in years. Find a formula for the GDP during the 15 years period.

Rate: $r(t) = (5 + 4 \cos(t/2)) / 100$.

GDP: $G = G_0 e^{(\int r(t)\, dt)} = G_0 e^{(0.05t + 8 \sin(t/2))}$.

Answer: $G = G_0 e^{(0.05t + 8 \sin(t/2))}$.

8. (a) Find the foci and the directrix of the ellipse $4x^2 + 16y^2 = 25$. Hence, identify the major axis of the ellipse where the foci are located.

$x^2/(25/4) + y^2/(25/16) = 1$.

$a = 5/2$, $b = 5/4$.

$e = \sqrt{(1 - (b/a)^2)} = 3/5$.

Foci: $(\pm ae, 0) = (\pm 3/2, 0)$.

Directrix: $x = \pm a/e = \pm 25/6$.

Major axis: x-axis.

Answer: Foci: $(\pm 3/2, 0)$, Directrix: $x = \pm 25/6$, Major axis: x-axis.

8. (b) (i) Find the equation of the tangents to the curve $y/x^3 = -1/8$ that will pass through the point (1, 1).

$y = (-1/8) x^3$.

$dy/dx = (-3/8) x^2$.

At $(a, (-1/8)a^3)$: Slope $= (-3/8)a^2$.

Tangent: $y + (1/8)a^3 = (-3/8)a^2 (x - a)$, passes through (1, 1).

Solve: a = 2, -4.

Tangents: $y = -3x + 4$, $y = 12x - 11$.

Answer: $y = -3x + 4$, $y = 12x - 11$.

(ii) Show that a curve defined by the parametric equations $x = (2/3)t + 7$ and $y = 5t - 1$ is a straight line.

Eliminate t: $t = (x - 7) / (2/3)$, $y = 5 ((x - 7) / (2/3)) - 1$.

$y = (15/2)x - 113/2$ (linear).

Answer: Proven.

8. (c) If $y^2 = 16 - (x - 1)^2$ is an equation of a circle, verify whether its radius is $\cos \theta \pm \sqrt{(\cos^2 \theta + 15)}$ where θ is an angle made by the radius and the polar axis.

Circle: $(x - 1)^2 + y^2 = 16$, radius = 4.

Given radius: $\cos \theta \pm \sqrt{(\cos^2 \theta + 15)} \neq 4$ (incorrect form).

Answer: Not verified.

8. (d) (i) Convert $r(1 - 2 \sin \theta) = 2$ into Cartesian form.

$r = x^2 + y^2$, $\sin \theta = y/r$.

$r - 2r \sin \theta = 2 \rightarrow \sqrt{(x^2 + y^2)} - 2y = 2$.

Square: $x^2 + y^2 = (2y + 2)^2$.

Answer: $x^2 + y^2 = (2y + 2)^2$.

(ii) Sketch the graph of $r = 1 + 2 \sin \theta$.

Cardioid: $r = 1$ at $\theta = 3\pi/2$, $r = 3$ at $\theta = \pi/2$, $r = 0$ at $\theta = 7\pi/6, 11\pi/6$.

Answer: Cardioid sketched.