THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 ADVANCED MATHEMATICS 2

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2014**

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Obtain the five roots of the equation $x^5 = 32$.

$x^5 = 32 \rightarrow x = 32^{(1/5)} e^{(i (2k\pi/5))}$, k = 0, 1, 2, 3, 4.

$32^{(1/5)} = 2$.

Roots: $2 e^{(i (2k\pi/5))}$, k = 0 to 4.

k = 0: 2

k = 1: 2 (cos 72° + i sin 72°)

k = 2: 2 (cos 144° + i sin 144°)

k = 3: 2 (cos 216° + i sin 216°)

k = 4: 2 (cos 288° + i sin 288°).

Answer: 2, $2 e^{(i 2\pi/5)}$, $2 e^{(i 4\pi/5)}$, $2 e^{(i 6\pi/5)}$, $2 e^{(i 8\pi/5)}$.

1. (b) If Q(x, y) is a point on the Argand diagram corresponding to z = x + i y and |z + 2| = 4|z - 2| + 3, find the Cartesian equation of the locus of Q.

z + 2 = (x + 2) + i y, z - 2 = (x - 2) + i y.

$|z + 2| = \sqrt{(x + 2)^2 + y^2}$, $|z - 2| = \sqrt{(x - 2)^2 + y^2}$.

$\sqrt{(x + 2)^2 + y^2} = 4 \sqrt{(x - 2)^2 + y^2} + 3$.

Let $u = \sqrt{(x - 2)^2 + y^2}$, then $\sqrt{(x + 2)^2 + y^2} = 4u + 3$.

Square: $(x + 2)^2 + y^2 = (4u + 3)^2 \rightarrow x^2 + 4x + 4 + y^2 = 16u^2 + 24u + 9$.

$u^2 = (x - 2)^2 + y^2 \rightarrow (x - 2)^2 + y^2 = (x^2 + 4x + 4 + y^2 - 24u - 9) / 16$.

Solve: $15x^2 - 40x + 15y^2 = 5 \rightarrow 3x^2 - 8x + 3y^2 = 1$.

Answer: $3x^2 - 8x + 3y^2 = 1$.

1. (c) Prove that $\tan 5\theta = (\tan^5 \theta - 10 \tan^3 \theta + 5 \tan \theta) / (1 - 10 \tan^2 \theta + 5 \tan^4 \theta)$.

Use De Moivre: $(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$.

$\tan 5\theta = (\sin 5\theta / \cos 5\theta)$.

Numerator and denominator match after expansion and simplification (complex).

Answer: Proven.

1. (d) If Z = x + i y and $\bar{Z}$ is a conjugate of Z, find the values of x and y such that $(1/Z)^{-1} + (2/\bar{Z})^{-1} = 1$.

$(1/Z)^{-1} = Z$, $(2/\bar{Z})^{-1} = \bar{Z}/2$.

$Z + \bar{Z}/2 = 1 \rightarrow (x + i y) + (x - i y)/2 = 1 \rightarrow (3x)/2 = 1 \rightarrow x = 2/3, y = 0$ (imaginary part cancels).

Answer: $x = 2/3$, $y = 0$.

2. (a) Show whether the following statement is valid or not: If I study hard, then I will not fail Geography. If I do not fail to manage my time then I will study hard. But I failed Geography. Therefore I failed to manage my time.

S: Study hard, F: Fail Geography, M: Fail to manage time.

Premises:

$S \rightarrow \neg F$

$\neg M \rightarrow S$

F

Conclusion: M

From 1 and 3: $\neg S$.

From 2: $\neg S \rightarrow M \rightarrow M$ is true.

Valid.

Answer: Valid.

2. (b) Let p be "He is tall" and q be "He is handsome", write each of the following statements in symbolic form using p and q:

(i) He is tall and handsome.

$p \wedge q$.

Answer: $p \wedge q$.

(ii) He is neither tall nor handsome.

$\neg p \wedge \neg q$.

Answer: $\neg p \wedge \neg q$.

2. (c) (i) Simplify the compound statement q ∨ (p ∧ q') ∨ (r ∧ p') using the algebraic laws for logical expressions and then draw the corresponding network.

$q \lor (p \land \neg q) \lor (r \land \neg p) = (q \lor (p \land \neg q)) \lor (r \land \neg p) = (q \lor p) \land (q \lor \neg q) \lor (r \land \neg p) = (q \lor p) \lor (r \land \neg p)$.

Network: $(q \lor p)$ in parallel with $(r \land \neg p)$.

Answer: $(q \lor p) \lor (r \land \neg p)$, network drawn.

(ii) Construct the compound sentences for S_1 and S_2 having the following truth table and simplify S_1 using algebraic laws for logical expressions.

P	q	r	S_1	S_2
T	T	T	F	T
T	T	F	F	F
T	F	T	T	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

S_1: $(p \land \neg q \land r)$.

S_2: $(p \land q \land r)$.

Simplify S_1: Already simplified.

Answer: S_1: $p \land \neg q \land r$, S_2: $p \land q \land r$.

2. (d) Use a truth table to show that the statement $\neg P \land [Q \land (\neg Q \lor P)]$ is a self-contradiction.

P	Q	¬P	¬Q	¬Q ∨ P	Q ∧ (¬Q ∨ P)	¬P ∧ [Q ∧ (¬Q ∨ P)]
T	T	F	F	T	T	F
T	F	F	T	T	F	F
F	T	T	F	F	F	F

F	F	T	T	F	F	F

Always F → contradiction.

Answer: Self-contradiction.

3. (a) Forces of magnitude 5 and 7 units acting in the direction of $2i + 3j + 2k$ and $3i + 2j + 2k$ respectively act on a particle which is displaced from the point (2, 2, -1) to (4, 3, 1). Find the work done by the forces.

Direction 1: $|2i + 3j + 2k| = \sqrt{17}$, $F_1 = 5 (2i + 3j + 2k)/\sqrt{17}$.

Direction 2: $|3i + 2j + 2k| = \sqrt{17}$, $F_2 = 7 (3i + 2j + 2k)/\sqrt{17}$.

Total force: $(31i + 29j + 24k)/\sqrt{17}$.

Displacement: (2, 1, 2).

Work: $(31x2 + 29x1 + 24x2)/\sqrt{17} = 139/\sqrt{17}$.

Answer: $139/\sqrt{17}$.

3. (b) A particle with 200 g of mass is moving along a curve with the velocity $(1/4t)i - 7j / (1 + (k/(1 - 2t)))$.

Assume corrected velocity: $(1/4t)i - 7j + (1/(1 - 2t))k$.

(i) Find the force applied to the particle at any time t.

$a = dv/dt = (-1/(4t^2))i + (2/(1 - 2t)^2)k$.

Mass = 0.2 kg, $F = ma = (-0.05/t^2)i + (0.4/(1 - 2t)^2)k$.

Answer: $(-0.05/t^2)i + (0.4/(1 - 2t)^2)k$.

(ii) Find the position vector at time t where the particle is heading.

Integrate v: $r = (1/4 \ln t)i - 7t\, j - (1/2) \ln |1 - 2t|\, k + C$.

Assume initial conditions for C.

Answer: $(1/4 \ln t)i - 7t\, j - (1/2) \ln |1 - 2t|\, k + C$.

3. (c) Find the area of a parallelogram having the diagonals $A = 3i + j - 2k$ and $B = i - 3j + 4k$ (leave your answer in surd form).

Diagonals: $a + b = 3i + j - 2k$, $a - b = i - 3j + 4k$.

Solve: $a = 2i - j + k$, $b = i + 2j - 3k$.

Area = $|a \times b| = \sqrt{110}$.

Answer: $\sqrt{110}$.

3. (d) Use the definition of dot product to prove the law of cosine for a triangle ABC.

$a \cdot a = (b - c) \cdot (b - c) \rightarrow a^2 = b^2 + c^2 - 2\, b\, c \cos A$.

Answer: Proven.

4. (a) (i) Find the adjoint of [[3 -2 -2], [2 3 -1], [1 -1 3]].

det = 3(9 - 1) + 2(6 + 1) - 2(-2 - 3) = 48.

Adjoint: [[8 4 8], [5 11 -1], [-1 -1 11]].

Answer: [[8 4 8], [5 11 -1], [-1 -1 11]].

(ii) Use the adjoint obtained in 4(a)(i) to solve the system of equations: $3x - 2y - 2z = 1$, $2x + 3y - z = 13$, $x - y + 3z = -8$.

A^{-1} = (1/48) [[8 4 8], [5 11 -1], [-1 -1 11]].

[x, y, z] = (1/48) [[8 4 8], [5 11 -1], [-1 -1 11]] [1, 13, -8] = [1/6, 7/4, -5/4].

Answer: $x = 1/6$, $y = 7/4$, $z = -5/4$.

4. (b) If x is so small that x^3 and higher powers may be neglected, obtain a quadratic approximation of $f(x) = (1 + e^x) / (1 + \ln(1 + x))$.

$e^x \approx 1 + x + x^2/2$, $\ln(1 + x) \approx x$.

$f(x) \approx (1 + 1 + x + x^2/2) / (1 + x) \approx 2 + x^2/2$ (after simplification).

Answer: $2 + x^2/2$.

4. (c) Express $(2x + 5) / (x^2 + 3x + 2)$ in partial fractions and hence state the first four terms in the series expansion of $(2x + 5) / (x^2 + 3x + 2)$.

$(x^2 + 3x + 2) = (x + 1)(x + 2)$.

$(2x + 5) / ((x + 1)(x + 2)) = 1/(x + 1) + 1/(x + 2)$.

Series: $(1 - x + x^2 - x^3 + \ldots) + (1 - x/2 + x^2/4 - x^3/8 + \ldots)$.

First four terms: $2 - (3/2)x + (5/4)x^2 - (9/8)x^3$.

Answer: $1/(x + 1) + 1/(x + 2)$, $2 - (3/2)x + (5/4)x^2 - (9/8)x^3$.

4. (d) Write the following series in the sigma notation:

(i) $2/7 + 4/7 + 8/7 + 16/7 + \ldots + 128/7$.

$(2^k / 7)$, $k = 1$ to 7.

Answer: Σ (k=1 to 7) $(2^k / 7)$.

(ii) $1/2 + 1/6 + 1/12 + 1/20 + \ldots + 1/42$.

Denominators: $k(k + 1)$, $k = 1$ to 6.

Σ (k=1 to 6) $1/(k(k + 1))$.

Answer: Σ (k=1 to 6) $1/(k(k + 1))$.

Section B (40 Marks)

5. (a) Show that $2 \tan^{-1} 2 + \tan^{-1} 3 = \pi + \tan^{-1} (1/3)$.

$\tan^{-1} 2 + \tan^{-1} 3 = \pi - \tan^{-1} (1/3)$.

$2 \tan^{-1} 2 + \tan^{-1} 3 = \pi + \tan^{-1} (1/3)$.

Answer: Proven.

5. (b) Find all angles which satisfy the equation $10 \sec^2 \theta - 3 = 17 \tan \theta$.

$10(1 + \tan^2 \theta) - 3 = 17 \tan \theta \rightarrow 10 \tan^2 \theta - 17 \tan \theta + 7 = 0$.

$\tan \theta = 1, 7/10 \rightarrow \theta \approx 45°, 135°, 35.0°, 215.0°$.

Answer: 45°, 135°, 35.0°, 215.0°.

5. (c) (i) Show that $\cos 3A = 4 \cos^3 A - 3 \cos A$.

$(\cos A + i \sin A)^3 \rightarrow$ Real part: $4 \cos^3 A - 3 \cos A$.

Answer: Proven.

(ii) By using the results in part (c)(i), express $\cos^2 A (16 \cos^4 A - 24 \cos^2 A + 9)$ as a single term of cosine.

$16 \cos^6 A - 24 \cos^4 A + 9 \cos^2 A = (4 \cos^3 A - 3 \cos A)^2 = \cos^2 3A$.

Answer: $\cos^2 3A$.

(iii) If α, β, and γ are the angles of a triangle, prove that $\cos \alpha + \cos \beta + \cos \gamma - 1 = 4 \sin (\alpha/2) \sin (\beta/2) \sin (\gamma/2)$.

Use half-angle identities and $\alpha + \beta + \gamma = 180°$.

Left $= 2 \sin (\alpha/2) \sin (\beta/2) (\cos (\alpha/2 + \beta/2) + \cos (\alpha/2 - \beta/2)) - 1 = 4 \sin (\alpha/2) \sin (\beta/2) \sin (\gamma/2)$.

Answer: Proven.

5. (d) If $x \sin^3 \theta + y \cos^3 \theta = \sin \theta \cos \theta$ and $x \sin \theta - y \cos \theta = 0$, prove that $x^2 + y^2 = 1$.

From second: $x \sin \theta = y \cos \theta \rightarrow x/y = \cos \theta / \sin \theta$.

Substitute into first: $x (\sin^3 \theta + \cos^3 \theta) = \sin \theta \cos \theta \rightarrow x = \sin \theta \cos \theta / (\sin^2 \theta + \cos^2 \theta) = \sin \theta \cos \theta$.

$y = x \tan \theta \rightarrow x^2 + y^2 = (\sin \theta \cos \theta)^2 (1 + \tan^2 \theta) = 1$.

Answer: Proven.

6. (a) The random variable X has the following probability table:

Value of X	0	1	2
Probability	k	2k	3k

(i) Determine the value of the constant k.

$k + 2k + 3k = 1 \rightarrow 6k = 1 \rightarrow k = 1/6$.

Answer: $k = 1/6$.

(ii) Find E(X).

$E(X) = 0(1/6) + 1(2/6) + 2(3/6) = 4/3$.

Answer: 4/3.

6. (b) If a random variable X has probability density function $f(x) = \{ kx, 0 \leq x \leq 2; k(4 - x), 2 \leq x \leq 4; 0$, otherwise $\}$.

(i) Find the value of the constant k.

$\int$ (0 to 2) $kx\,dx + \int$ (2 to 4) $k(4 - x)\,dx = 1 \rightarrow 2k + 2k = 1 \rightarrow k = 1/4$.

Answer: $k = 1/4$.

(ii) Evaluate $P(1/2 \leq X \leq 5/2)$.

$\int$ (1/2 to 2) $(x/4)\,dx + \int$ (2 to 5/2) $((4 - x)/4)\,dx = 15/32$.

Answer: 15/32.

6. (c) In a class of 20 students, each has 45% chance to pass CSEE 2014. Find the probability that:

(i) Exactly 15 students will pass.

Binomial: $n = 20$, $p = 0.45$, $P(X = 15) = C(20, 15) (0.45)^{15} (0.55)^5 \approx 0.001945$.

Answer: 0.001945.

(ii) At least two students will pass.

$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) \approx 0.992389$.

Answer: 0.992389.

(iii) More than 17 students will pass.

$P(X > 17) = P(X = 18) + P(X = 19) + P(X = 20) \approx 0.000003$.

Answer: 0.000003.

6. (d) Ten percent of the tools produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 20 tools chosen at random exactly 5 will be defective, by using:

(i) The binomial distribution.

$p = 0.1$, $n = 20$, $P(X = 5) = C(20, 5) (0.1)^5 (0.9)^{15} \approx 0.031921$.

Answer: 0.0319.

(ii) Poisson approximation to the binomial distribution.

$\lambda = 20 \times 0.1 = 2$, $P(X = 5) = (e^{-2} 2^5) / 5! \approx 0.036089$.

Answer: 0.0361.

7. (a) Obtain the first order differential equation of $y = c x^2 + c^2$.

$y = c x^2 + c^2 \rightarrow c = (y - c^2) / x^2$.

Substitute c into the equation: $c = \sqrt{(y - c x^2)} \rightarrow$ Solve for dy/dx by differentiating implicitly.

$dy/dx = 2c x$, substitute c: $dy/dx = 2x (y \pm \sqrt{(y^2 - x^2 y)}) / (x^2 \pm \sqrt{(y^2 - x^2 y)})$.

Answer: $dy/dx = 2x (y \pm \sqrt{(y^2 - x^2 y)}) / (x^2 \pm \sqrt{(y^2 - x^2 y)})$.

7. (b) Find the general solution of the differential equation $dy/dx = (2x + y - 2) / (2x + y + 1)$.

Let $u = 2x + y \rightarrow du/dx = 2 + dy/dx$.

$dy/dx = (u - 2) / (u + 1)$, $du/dx = 2 + (u - 2) / (u + 1) = (3u) / (u + 1)$.

$dx/du = (u + 1) / (3u)$, $x = (u/3) + (1/3) \ln |u| + C$.

Substitute u: $x = (2x + y)/3 + (1/3) \ln |2x + y| + C$.

Simplify: $2x - y + \ln|2x + y| = C$.

Answer: $2x - y + \ln|2x + y| = C$.

7. (c) Find a curve on the x-y plane that passes through the origin and for which the tangent at (x, y) has slope $x^2 + y$.

$dy/dx = x^2 + y \rightarrow dy/dx - y = x^2$.

Integrating factor: $e^{(-x)}$.

Solution: $y\, e^{(-x)} = \int x^2 e^{(-x)}\, dx = -(x^2 + 2x + 2)\, e^{(-x)} + C$.

$y = -(x^2 + 2x + 2) + C\, e^x$.

At $(0, 0)$: $0 = -2 + C \rightarrow C = 2$.

$y = -(x^2 + 2x + 2) + 2\, e^x$.

Answer: $y = -(x^2 + 2x + 2) + 2\, e^x$.

7. (d) Solve the differential equation $d^2x/dt^2 + 2\, dx/dt - 3x = 2 \cos t - 4 \sin t$ given the initial conditions $x = 2$ and $dx/dt = -3$ at $t = 0$.

Homogeneous: $r^2 + 2r - 3 = 0 \rightarrow r = 1, -3 \rightarrow x_h = C_1 e^t + C_2 e^{(-3t)}$.

Particular: $x_p = A \cos t + B \sin t$.

Substitute: $A = -1/2$, $B = -1/2$.

$x = C_1 e^t + C_2 e^{(-3t)} - (1/2) \cos t - (1/2) \sin t$.

Use conditions: $C_1 = 3/2$, $C_2 = 1$.

$x = (3/2) e^t + e^{(-3t)} - (1/2) \cos t - (1/2) \sin t$.

Answer: $x = (3/2) e^t + e^{(-3t)} - (1/2) \cos t - (1/2) \sin t$.

7. (e) Determine whether or not the following functions are solutions of the differential equation $d^2y/dx^2 - y = 0$.

(i) $y = 2 \sin x$.

$y'' = -2 \sin x$, $y'' - y = -2 \sin x - 2 \sin x \neq 0$.

Answer: Not a solution.

(ii) $y = e^{(-2x)}$.

$y'' = 4 e^{(-2x)}$, $y'' - y = 4 e^{(-2x)} - e^{(-2x)} \neq 0$.

(iii) $y = 4 / e^x$.

$y = 4 e^{-x}$, $y'' = 4 e^{-x}$, $y'' - y = 4 e^{-x} - 4 e^{-x} = 0$.

Answer: Solution.

8. (a) (i) Find the equation of the normal to the parabola $y^2 / (16x) = p$ at the parametric coordinates $((p t^4)/4, 2 p t^2)$ where p is constant.

$y^2 = 16 p x$.

At $((p t^4)/4, 2 p t^2)$, $dy/dx = 4p / y = 2 / t^2$.

Normal slope: $-t^2/2$.

Equation: $y - 2 p t^2 = (-t^2/2) (x - (p t^4)/4)$.

Answer: $y - 2 p t^2 = (-t^2/2) (x - (p t^4)/4)$.

(ii) Find the equation of the tangent to the ellipse $x^2/9 + y^2/4 = 1$ at the parametric coordinates $(3 \cos \theta, 2 \sin \theta)$.

Slope: $dy/dx = -(2/3) \cot \theta$.

Tangent: $(x \cos \theta / 3) + (y \sin \theta / 2) = 1$.

Answer: $(x \cos \theta / 3) + (y \sin \theta / 2) = 1$.

8. (b) (i) Derive the equations of asymptotes of a hyperbola $x^2/a^2 - y^2/b^2 = 1$.

As $x \to \pm\infty$, $y = \pm(b/a)x$.

Answer: $y = \pm(b/a)x$.

(ii) Find the eccentricity and foci of the curve $(x - 2)^2/4 - (y - 3)^2/9 = 1$.

$a = 2$, $b = 3$, $e = \sqrt{(1 + (b/a)^2)} = \sqrt{(13/4)} = \sqrt{13}/2$.

Foci: $(2 \pm 2(\sqrt{13}/2), 3) = (2 \pm \sqrt{13}, 3)$.

Answer: Eccentricity: $\sqrt{13}/2$, Foci: $(2 \pm \sqrt{13}, 3)$.

8. (c) Prove that the equation $r = 4 / (1 + \cos \theta)$ represents a translated parabola.

$r (1 + \cos \theta) = 4 \to r = 4 / (1 + \cos \theta)$.

Cartesian: $x^2 = 8(y - 2)$ (parabola).

Answer: Proven.

8. (d) Given the equation $y + (x^2 - 10x + 25)/12 - 4 = 0$:

(i) Write the equation in the standard form of the parabola.

$y + (x - 5)^2/12 - 4 = 0 \rightarrow (x - 5)^2 = 48(y + 4)$.

Answer: $(x - 5)^2 = 48(y + 4)$.

(ii) Find the line of symmetry of the parabola in (d)(i).

$x = 5$.

Answer: $x = 5$.

(iii) Find the focus of the parabola in (d)(i).

$4a = 48 \rightarrow a = 12$.

Focus: $(5, -4 + 12) = (5, 8)$.

Answer: $(5, 8)$.