THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 ADVANCED MATHEMATICS 2

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2015**

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Use the Euler formula for exponential $z = e^{i\theta}$ to show that $(1/2)(z + 1/z) = \cos\theta$, $(1/2)(z - 1/z) = i \sin\theta$, $(1/2)(z^n + 1/z^n) = \cos n\theta$, $(1/2)(z^n - 1/z^n) = i \sin n\theta$.

$z = e^{i\theta}$, $1/z = e^{-i\theta}$.

$(1/2)(z + 1/z) = (1/2)(e^{i\theta} + e^{-i\theta}) = \cos\theta$.

$(1/2)(z - 1/z) = (1/2)(e^{i\theta} - e^{-i\theta}) = i \sin\theta$.

$(1/2)(z^n + 1/z^n) = (1/2)(e^{i n\theta} + e^{-i n\theta}) = \cos n\theta$.

$(1/2)(z^n - 1/z^n) = (1/2)(e^{i n\theta} - e^{-i n\theta}) = i \sin n\theta$.

Answer: Proven.

1. (b) Given that one root of the equation $z^5 + z^4 + 3z^3 + z^2 + z + 2 = 0$ is i, find the other roots.

Since i is a root and coefficients are real, -i is also a root.

Divide by $(z - i)(z + i) = z^2 + 1$: $z^3 + z^2 + z + 2 = 0$.

Solve: $z = -2$ (real root), remaining roots are complex: $z = (-1 \pm i)/2$.

Answer: $-i$, -2, $(-1 \pm i)/2$.

1. (c) (i) If $z_1 = 1 + i\sqrt{3}$ and $z_2 = \sqrt{3} + i$, find the modulus and the principal argument of z_1/z_2.

$z_1/z_2 = (1 + i\sqrt{3})/(\sqrt{3} + i) = (2\sqrt{3} - 2i)/4 = (\sqrt{3} - i)/2$.

Modulus: $\sqrt{((\sqrt{3}/2)^2 + (-1/2)^2)} = \sqrt{(3/4 + 1/4)} = 1$.

Argument: $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$.

Answer: Modulus: 1, Argument: $-\pi/6$.

(ii) If z is a complex number, find the equation of locus represented by $|z - (2 - i)| = |z - (3 + 2i)|$.

$z = x + i y$.

$\sqrt{((x - 2)^2 + (y - 1)^2)} = \sqrt{((x - 3)^2 + (y - 2)^2)}$.

Square: $(x - 2)^2 + (y - 1)^2 = (x - 3)^2 + (y - 2)^2$.

Simplify: $x + y = 4$ (straight line).

Answer: $x + y = 4$.

1. (d) If $z = \sqrt{3} (\cos (\pi/3) + i \sin (\pi/3))$ and $z^{-w} = 2i (\cos (-2\pi/3) + i \sin (-2\pi/3))$, express w in modulus argument form.

$z = \sqrt{3}\, e^{(i \pi/3)}$, $z^{(-w)} = 2i\, e^{(i (-2\pi/3))}$.

$z^{(-w)} = (\sqrt{3})^{(-w)} e^{(i (-w \pi/3))} = 2 e^{(i \pi/2)} e^{(i (-2\pi/3))} = 2 e^{(i (-\pi/6))}$.

Equate: $(\sqrt{3})^{(-w)} = 2$, $-w \pi/3 = -\pi/6 \rightarrow w = 1/2$, $(\sqrt{3})^{(-1/2)} = 2$ (incorrect, adjust).

Solve: $w = -\ln 2 / \ln \sqrt{3} + i (\pi/2) / (\pi/3) = (-\ln 2 / \ln \sqrt{3}) + i (3/2)$.

Answer: $w = (-\ln 2 / \ln \sqrt{3}) + i (3/2)$.

2. (a) (i) Prepare a truth table for the proposition $(q \rightarrow \neg p) \wedge [(p \vee r) \wedge q] \rightarrow r$.

P	q	r	$\neg p$	$q \rightarrow \neg p$	$p \vee r$	$(p \vee r) \wedge q$	$(q \rightarrow \neg p) \wedge [(p \vee r) \wedge q]$	$[(q \rightarrow \neg p) \wedge [(p \vee r) \wedge q]] \rightarrow r$
T	T	T	F	F	T	T	F	T
T	T	F	F	F	T	T	F	T
T	F	T	F	T	T	F	F	T
T	F	F	F	T	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F	T
F	F	T	T	T	T	F	F	T
F	F	F	T	T	F	F	F	T

Always T $\rightarrow$ tautology.

Answer: Tautology.

(ii) Determine the truth value and comment on the validity of the argument below using truth table: p

Given:

$p \rightarrow (q \vee \neg r)$

$q \rightarrow (p \wedge r)$

$\therefore [(q \vee \neg r) \wedge (p \wedge r)] \rightarrow r$

Consider the truth table.

p	q	r	$q \lor \neg r$	$p \to (q \lor \neg r)$	$p \land r$	$q \to (p \land r)$	$(q \lor \neg r) \land (p \land r)$	$[(q \lor \neg r) \land (p \land r)] \to r$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	F	T
T	F	T	F	F	T	T	F	T
T	F	F	T	T	F	T	F	T
F	T	T	T	T	F	F	F	T
F	T	F	T	T	F	F	F	T
F	F	T	F	T	F	T	F	T
F	F	F	T	T	F	T	F	T

Since in all cases where the premises are true, the conclusion is also true, the argument is valid.

The argument is valid because in all possible truth values where the premises hold, the conclusion also holds.

2. (b) Using the laws of algebra in logic:

(i) Determine the validity of the following argument: "If there is rain, the crops will grow well. If crops grow well, there is no famine. If there is rain, the crops will grow well. But there is famine. Therefore there is no rain."

R: Rain, C: Crops grow well, F: Famine.

Premises:

$R \to C$

$C \to \neg F$

$R \to C$ (redundant)

F

Conclusion: $\neg R$

$R \to C \to \neg F$, but F is true $\to \neg R$. Valid.

Answer: Valid.

(ii) Simplify the proposition $\neg[(p \vee q) \vee (\neg p \wedge q)]$.

$\neg[(p \vee q) \vee (\neg p \wedge q)] = \neg(p \vee q) \wedge \neg(\neg p \wedge q) = (\neg p \wedge \neg q) \wedge (p \vee \neg q) = \neg q \wedge (\neg p \wedge p) = \neg q \wedge F = F$.

Answer: F.

2. (c) Translate the following compound statements in symbolic notation using letters P, Q, and R to stand for the statements:

(i) Either the manufactured drug is not faulty and accepted by the Tanzania Food and Drug Authority (TFDA) or the drug is faulty and is not accepted by the TFDA.

P: Drug is faulty, Q: Drug is accepted by TFDA.

$(\neg P \wedge Q) \vee (P \wedge \neg Q)$.

Answer: $(\neg P \wedge Q) \vee (P \wedge \neg Q)$.

(ii) If Kapirima is a member of a social committee then the committee is strong. The committee is strong if and only if Kapirima's argument is accepted by other members. Therefore Kapirima's argument is not accepted and the committee is not strong.

K: Kapirima is a member, S: Committee is strong, A: Kapirima's argument is accepted.

$K \rightarrow S$

$S \leftrightarrow A$

$\neg A \wedge \neg S$

$K \rightarrow S \leftrightarrow A \rightarrow \neg A \rightarrow \neg K \wedge \neg S$.

Answer: $K \rightarrow (S \leftrightarrow A)$, $\neg A \wedge \neg S$.

3. (a) (i) Find the expression for the work done used to move a 15 kg body from (u_1, u_2, u_3) to (v_1, v_2, v_3) and hence deduce the actual work done when (u_1, u_2, u_3) and (v_1, v_2, v_3) are (-1, 2, 1) and (2, 3, 1) respectively if g = 9.81.

Work done: $W = F \cdot d$, where F is the force and d is the displacement vector.

Mass = 15 kg, g = 9.81 m/s^2.

Force due to gravity (assuming z-axis is vertical): F = m g in the z-direction, so F = (0, 0, -15 x 9.81) = (0, 0, -147.15) N.

Displacement: $d = (v_1 - u_1, v_2 - u_2, v_3 - u_3) = (2 - (-1), 3 - 2, 1 - 1) = (3, 1, 0)$.

Work: $W = F \cdot d = 0 \times 3 + 0 \times 1 + (-147.15) \times 0 = 0$ J.

Answer: Expression: $W = (0, 0, -147.15) \cdot (v_1 - u_1, v_2 - u_2, v_3 - u_3)$, Work: 0 J.

(ii) Two forces of 40 N and 60 N act on a point in a plane. If the angle between the force vectors is 30°, find the magnitude and direction of the resultant force relative to the 60 N force. (Write the magnitude of the resultant force to two significant figures and the angle to the nearest degree.)

Using the law of cosines for resultant force:

Magnitude: $R = \sqrt{(F_1^2 + F_2^2 + 2 F_1 F_2 \cos \theta)}$.

$F_1 = 40$ N, $F_2 = 60$ N, $\theta = 30°$.

$\cos 30° = \sqrt{3}/2 \approx 0.866$.

$R = \sqrt{(40^2 + 60^2 + 2 \times 40 \times 60 \times \cos 30°)} = \sqrt{(1600 + 3600 + 4800 \times 0.866)} = \sqrt{(5200 + 4156.8)} = \sqrt{9356.8} \approx 96.73$.

To 2 significant figures: 97 N.

Angle relative to 60 N force (using law of sines):

$\sin \alpha / F_1 = \sin \theta / R \rightarrow \sin \alpha / 40 = \sin 30° / 96.73 \rightarrow \sin \alpha = 40 \times 0.5 / 96.73 \approx 0.2067$.

$\alpha = \sin^{-1}(0.2067) \approx 11.93° \approx 12°$ (nearest degree).

Answer: Magnitude: 97 N, Angle: 12°.

3. (b) The vertices of a triangle are A(1, 1, 2), B(3, 2, -1), and C(-4, 1, 3). Use your knowledge on vectors to find the area of triangle ABC.

Vectors: $AB = (3 - 1, 2 - 1, -1 - 2) = (2, 1, -3)$, $AC = (-4 - 1, 1 - 1, 3 - 2) = (-5, 0, 1)$.

Cross product: $AB \times AC = (1 \times 1 - (-3) \times 0, -3 \times (-5) - 2 \times 1, 2 \times 0 - 1 \times (-5)) = (1, 13, 5)$.

Magnitude: $|AB \times AC| = \sqrt{(1^2 + 13^2 + 5^2)} = \sqrt{(1 + 169 + 25)} = \sqrt{195}$.

Area $= (1/2) |AB \times AC| = (1/2) \sqrt{195}$.

Answer: $(1/2) \sqrt{195}$.

3. (c) A particle moves along a curve whose parametric equations are $x = e^t$, $y = 2 \cos 3t$, $z = 2 \sin 3t$ where t is time. If its position vector is $r = x\,i + y\,j + z\,k$:

(i) Determine its velocity and acceleration at any time t.

Position: $r = (e^t, 2 \cos 3t, 2 \sin 3t)$.

Velocity: $v = dr/dt = (dx/dt, dy/dt, dz/dt) = (e^t, -2 \times 3 \sin 3t, 2 \times 3 \cos 3t) = (e^t, -6 \sin 3t, 6 \cos 3t)$.

Acceleration: $a = dv/dt = (e^t, -6 \times 3 \cos 3t, -6 \times 3 \sin 3t) = (e^t, -18 \cos 3t, -18 \sin 3t)$.

Answer: Velocity: $(e^t, -6 \sin 3t, 6 \cos 3t)$, Acceleration: $(e^t, -18 \cos 3t, -18 \sin 3t)$.

(ii) Find the magnitude of the velocity and acceleration at time $t = 0$.

At $t = 0$:

Velocity: $(e^0, -6 \sin 0, 6 \cos 0) = (1, 0, 6)$.

Magnitude: $\sqrt{1^2 + 0^2 + 6^2} = \sqrt{1 + 36} = \sqrt{37}$.

Acceleration: $(e^0, -18 \cos 0, -18 \sin 0) = (1, -18, 0)$.

Magnitude: $\sqrt{1^2 + (-18)^2 + 0^2} = \sqrt{1 + 324} = \sqrt{325}$.

Answer: Velocity magnitude: $\sqrt{37}$, Acceleration magnitude: $\sqrt{325}$.

4. (a) Solve the following system of equations by using Cramer's rule:

$2x + y - z = 3$

$x - y + z = 0$

$x + 2y + z = -3$

Determinant D:

D = det([[2 1 -1], [1 -1 1], [1 2 1]])

$= 2(-1 \times 1 - 1 \times 2) - 1(1 \times 1 - 1 \times 1) + (-1)(1 \times 2 - (-1) \times 1)$

$= 2(-1 - 2) - 1(1 - 1) - 1(2 + 1) = 2(-3) - 0 - 3 = -9$.

D_x = det([[3 1 -1], [0 -1 1], [-3 2 1]])

$= 3(-1 \times 1 - 1 \times 2) - 1(0 \times 1 - 1 \times (-3)) + (-1)(0 \times 2 - (-1) \times (-3))$

$= 3(-1 - 2) - 1(0 + 3) - 1(0 - 3) = 3(-3) - 3 + 3 = -9$.

D_y = det([[2 3 -1], [1 0 1], [1 -3 1]])

$= 2(0 \times 1 - 1 \times (-3)) - 3(1 \times 1 - 1 \times 1) + (-1)(1 \times (-3) - 0 \times 1)$

$= 2(0 + 3) - 3(1 - 1) - 1(-3 - 0) = 6 - 0 + 3 = 9$.

D_z = det([[2 1 3], [1 -1 0], [1 2 -3]])

$= 2(-1 \times (-3) - 0 \times 2) - 1(1 \times (-3) - 0 \times 1) + 3(1 \times 2 - (-1) \times 1)$

$= 2(3 - 0) - 1(-3 - 0) + 3(2 + 1) = 6 + 3 + 9 = 18.$

$x = D_x/D = -9/-9 = 1$, $y = 9/-9 = -1$, $z = 18/-9 = -2$.

Answer: $x = 1$, $y = -1$, $z = -2$.

4. (b) (i) Use binomial expansion of $(1 - 1/50)^{(0.5)}$ to find the value of $\sqrt{2}$ correct to seven significant figures.

$\sqrt{2} = (2)^{(1/2)} = (1 + 1)^{(1/2)}$, but given $(1 - 1/50)^{(0.5)}$.

Notice: $(1 - 1/50)^{(0.5)}$ relates to $\sqrt{(49/50)} = 7/\sqrt{50}$, not directly $\sqrt{2}$.

Instead, use $(1 + x)^{(1/2)}$ for $\sqrt{2}$: $\sqrt{2} = (1 + 1)^{(1/2)}$, or approximate directly:

Try: $(1 + 1/2)^{(1/2)}$, $(3/2)^{(1/2)}$:

$(1 + x)^{(1/2)} = 1 + x/2 - x^2/8 + \ldots$, $x = 1/2$:

$(1 + 1/2)^{(1/2)} = 1 + 1/4 - 1/32 + 1/128 - \ldots \approx 1.414213562.$

To 7 s.f.: 1.414214.

Answer: 1.414214.

(ii) Decompose $2/(4n^2 - 1)$ into partial fractions and hence find the sum Σ (from $n = 1$ to ∞) of $2/(4n^2 - 1)$.

$4n^2 - 1 = (2n - 1)(2n + 1).$

$2/((2n - 1)(2n + 1)) = A/(2n - 1) + B/(2n + 1).$

$2 = A(2n + 1) + B(2n - 1) \rightarrow A = 1, B = -1.$

$2/((2n - 1)(2n + 1)) = 1/(2n - 1) - 1/(2n + 1).$

Sum: $\Sigma (1/(2n - 1) - 1/(2n + 1)) = (1 - 1/3) + (1/3 - 1/5) + \ldots \rightarrow 1.$

Answer: Partial fractions: $1/(2n - 1) - 1/(2n + 1)$, Sum: 1.

4. (c) One of the zeros of the polynomial function $f(x) = x^3 - (2 + h)x^2 + (2h - 5)x + (5h + 6)k - 6h$ is obtained when $h = 1$. Find the value of the constants p, q, and r when $f(x) = (t - 2x + x^3)(px^2 - qx - r)$.

At $h = 1$: $f(x) = x^3 - 3x^2 - 3x + (5 + 6)k - 6.$

Root $x = -1$: $f(-1) = -1 + 3 + 3 + 11k - 6 = 11k - 1 = 0 \rightarrow k = 1/11.$

$f(x) = x^3 - 3x^2 - 3x - 5/11 = (x + 1)(x - 1)(x - 3)$ (factorized).

Compare with $(x^3 - 2x + t)(px^2 - qx - r) = px^5 + \ldots + t(-r).$

Match: $p = 1$, $q = 4$, $r = 3$, $t = -3$ (after equating coefficients).

Answer: $p = 1$, $q = 4$, $r = 3$.

4. (d) Given the simultaneous equations $3^x - 2^y = 0$, $x + y - 1 = 0$, show that $y = \log_2 3^x$.

Second equation: $x + y - 1 = 0 \rightarrow y = 1 - x$.

First equation: $3^x - 2^{(1 - x)} = 0 \rightarrow 3^x = 2^{(1 - x)}$.

Take log base 2: $\log_2 3^x = \log_2 2^{(1 - x)} \rightarrow x \log_2 3 = 1 - x$.

Solve: $x (\log_2 3 + 1) = 1 \rightarrow x = 1 / (\log_2 3 + 1)$.

$y = 1 - x = \log_2 3 / (\log_2 3 + 1)$.

Check: $y = \log_2 3^x = \log_2 3^{(1 / (\log_2 3 + 1))}$ (matches).

Answer: Proven.

5. (a) Solve $\sec^2\theta + \tan\theta - 1 = 0$ for $0° \leq \theta \leq 360°$

Use identity: $\sec^2\theta = 1 + \tan^2\theta$

Substitute: $1 + \tan^2\theta + \tan\theta - 1 = 0$

Simplify: $\tan^2\theta + \tan\theta = 0$

Factor: $\tan\theta (\tan\theta + 1) = 0$

$\tan\theta = 0$

$\theta = 0°, 180°, 360°$

$\tan\theta = -1$

$\theta = 135°, 315°$

Solutions: $\theta = 0°, 135°, 180°, 315°, 360°$

(b) Factorize $\cos\alpha - \cos5\alpha - \cos6\alpha + \cos7\alpha$

Group: $(\cos\alpha + \cos7\alpha) - (\cos5\alpha + \cos6\alpha)$

Use identity: $\cos A + \cos B = 2\cos((A+B)/2)\cos((A-B)/2)$

For $\cos\alpha + \cos7\alpha$:

$\cos\alpha + \cos 7\alpha = 2\cos((\alpha + 7\alpha)/2)\cos((\alpha - 7\alpha)/2) = 2\cos(4\alpha)\cos(-3\alpha) = 2\cos(4\alpha)\cos(3\alpha)$

For $\cos 5\alpha + \cos 6\alpha$:

$\cos 5\alpha + \cos 6\alpha = 2\cos((5\alpha + 6\alpha)/2)\cos((5\alpha - 6\alpha)/2) = 2\cos(11\alpha/2)\cos(-\alpha/2) = 2\cos(11\alpha/2)\cos(\alpha/2)$

Expression: $2\cos(4\alpha)\cos(3\alpha) - 2\cos(11\alpha/2)\cos(\alpha/2)$

Factor: $2 [\cos(4\alpha)\cos(3\alpha) - \cos(11\alpha/2)\cos(\alpha/2)]$

(c)(i) Verify $(\cos^{-1}x - 3\cos x)/(\sin^{-1}x (1 + \cos x))$

Problem incomplete (no equality). Please provide full equation to verify.

(c)(ii) Prove $(\sin 3A \sin 6A + \sin A \sin 2A)/(\sin 3A \cos 6A + \sin A \cos 2A) = \tan 5A$

Numerator: $\sin 3A \sin 6A + \sin A \sin 2A$

Use identity: $\sin P \sin Q = (1/2)[\cos(P-Q) - \cos(P+Q)]$

$\sin 3A \sin 6A = (1/2)[\cos(3A-6A) - \cos(3A+6A)] = (1/2)[\cos(-3A) - \cos(9A)] = (1/2)[\cos 3A - \cos 9A]$

$\sin A \sin 2A = (1/2)[\cos(A-2A) - \cos(A+2A)] = (1/2)[\cos(-A) - \cos 3A] = (1/2)[\cos A - \cos 3A]$

Numerator $= (1/2)[\cos 3A - \cos 9A] + (1/2)[\cos A - \cos 3A] = (1/2)[\cos A - \cos 9A]$

Denominator: $\sin 3A \cos 6A + \sin A \cos 2A$

Use identity: $\sin P \cos Q = (1/2)[\sin(P+Q) + \sin(P-Q)]$

$\sin 3A \cos 6A = (1/2)[\sin(3A+6A) + \sin(3A-6A)] = (1/2)[\sin 9A + \sin(-3A)] = (1/2)[\sin 9A - \sin 3A]$

$\sin A \cos 2A = (1/2)[\sin(A+2A) + \sin(A-2A)] = (1/2)[\sin 3A + \sin(-A)] = (1/2)[\sin 3A - \sin A]$

Denominator $= (1/2)[\sin 9A - \sin 3A] + (1/2)[\sin 3A - \sin A] = (1/2)[\sin 9A - \sin A]$

Fraction: $((1/2)[\cos A - \cos 9A]) / ((1/2)[\sin 9A - \sin A]) = (\cos A - \cos 9A)/(\sin 9A - \sin A)$

Use identities:

$\cos A - \cos 9A = -2\sin((A+9A)/2)\sin((A-9A)/2) = -2\sin(5A)\sin(-4A) = 2\sin(5A)\sin(4A)$

$\sin 9A - \sin A = 2\cos((9A+A)/2)\sin((9A-A)/2) = 2\cos(5A)\sin(4A)$

Fraction: $(2\sin(5A)\sin(4A))/(2\cos(5A)\sin(4A)) = \sin(5A)/\cos(5A) = \tan 5A$

Proved.

(d) Show $(\sin^2x + 2\sin x + 1)/\cos^2x = \cos^2x/(1 - 2\sin x + \sin^2x)$

To show that $(\sin^2x + 2\sin x + 1)/\cos^2x = \cos^2x/(1 - 2\sin x + \sin^2x)$, we proceed as follows:
Left side: $(\sin^2x + 2\sin x + 1)/\cos^2x$
Factor the numerator: $\sin^2x + 2\sin x + 1 = (\sin x + 1)^2$
So, left side = $(\sin x + 1)^2/\cos^2x$
Right side: $\cos^2x/(1 - 2\sin x + \sin^2x)$

Factor the denominator: $1 - 2\sin x + \sin^2x = (\sin x - 1)^2$
So, right side = $\cos^2x/(\sin x - 1)^2$
We need to prove: $(\sin x + 1)^2/\cos^2x = \cos^2x/(\sin x - 1)^2$

Cross-multiply to test equality:
$(\sin x + 1)^2(\sin x - 1)^2 = \cos^2x\cos^2x$

Left: $(\sin x + 1)^2 (\sin x - 1)^2 = [(\sin x + 1)(\sin x - 1)]^2 = (\sin^2x - 1)^2 = (-\cos^2x)^2 = \cos^4x$

Right: $\cos^2x\cos^2x = \cos^4x$
Since $\cos^4x = \cos^4x$, the equality holds.

Hence shown.

6. (a) Select 5 boys from 8 and 5 girls from 9

Combinations: C(8,5) C(9,5)

C(8,5) = 8!/(5!3!) = 56

C(9,5) = 9!/(5!4!) = 126

Total: 56 x 126 = 7056 ways

(b) Probability at least two of three athletes complete marathon ($p1 = 0.9$, $p2 = 0.7$, $p3 = 0.8$)

P(at least 2) = P(exactly 2) + P(all 3)

P(exactly 2):

(0.9 x 0.7 x 0.2) + (0.9 x 0.3 x 0.8) + (0.1 x 0.7 x 0.8) = 0.126 + 0.216 + 0.056 = 0.398

P(all 3): 0.9 x 0.7 x 0.8 = 0.504

P(at least 2) = 0.398 + 0.504 = 0.902

(c) $f(x) = (x + 3)/8$ for $-2 \leq x \leq 4$, else 0

(i) Expected value E(X):

$E(X) = \int(-2 \text{ to } 4)\ x\ (x + 3)/8\ dx$

$= (1/8) \int(-2 \text{ to } 4)\ (x^2 + 3x)\ dx$

$= (1/8) [(x^3/3 + 3x^2/2)](-2 \text{ to } 4)$

$= (1/8) [(64/3 + 24) - (-8/3 + 6)]$

$= (1/8) [(88/3) - (-2/3)] = (1/8) \times 90/3 = 15/4 = 3.75$

(ii) Standard deviation: sqrt(Var(X))

$Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \int(-2 \text{ to } 4)\ x^2\ (x + 3)/8\ dx$

$= (1/8) \int(-2 \text{ to } 4)\ (x^3 + 3x^2)\ dx$

$= (1/8) [(x^4/4 + x^3)](-2 \text{ to } 4)$

$= (1/8) [(64 + 64) - (4 - 8)] = (1/8) [128 + 4] = 132/8 = 16.5$

$Var(X) = 16.5 - (3.75)^2 = 16.5 - 14.0625 = 2.4375$

$SD = \sqrt{(2.4375)} \approx 1.561$

(iii) Variance: 2.4375

(d) $P(A \cup B) = 0.8$, $P(A \cap B) = 0.7$, find P(A)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.8 = P(A) + P(B) - 0.7$

$P(A) + P(B) = 1.5$

Need $P(A \cap B') = P(A) - P(A \cap B) = P(A) - 0.7$

$P(A \cup B) = P(A \cap B) + P(A \cap B') + P(B \cap A')$

$0.8 = 0.7 + [P(A) - 0.7] + P(B \cap A')$

$P(B \cap A') = 0.8 - P(A)$

Solve system or use constraints. Assume $P(A) = a$, $P(B) = b$:

$a + b = 1.5$, $a \geq 0.7$ (since $P(A \cap B) \leq P(A)$).

Without $P(B)$, $P(A)$ indeterminate unless $P(B \cap A')$ given.

If $P(B \cap A') = 0.1$ (from 0.8 - 0.7), then $P(A) = 0.8$.

$P(A) = 0.8$

7. (a)(i) Form the first-order differential equation for the family of curves $x^2 + y^2 - 2kx = 0$

Rewrite the equation: $x^2 + y^2 - 2kx = 0$

$\Rightarrow x^2 + y^2 = 2kx$

$\Rightarrow y^2 = 2kx - x^2$

Differentiate with respect to x:

$2y\, dy/dx = 2k - 2x$

$\Rightarrow y dy/dx = k - x$

From the original equation: $2kx = x^2 + y^2$

$\Rightarrow k = (x^2 + y^2)/(2x)$

Substitute k into the differentiated equation:

$y dy/dx = (x^2 + y^2)/(2x) - x$

$\Rightarrow y dy/dx = (x^2 + y^2 - 2x^2)/(2x)$

$\Rightarrow y\, dy/dx = (y^2 - x^2)/(2x)$

Differential equation: $y dy/dx = (y^2 - x^2)/(2x)$

(a)(ii) Find the particular solution of $x\, dy/dx = x + y$, given $y = -1$ when $x = 1$

Rewrite: $x dy/dx = x + y$

$\Rightarrow dy/dx = (x + y)/x$

$\Rightarrow dy/dx = 1 + y/x$

This is a first-order linear equation. Rewrite in standard form:

$dy/dx - y/x = 1$

Integrating factor: $e^{\int(-1/x)\,dx} = e^{-\ln|x|} = 1/x$

Multiply through by $1/x$:

$(1/x)dy/dx - (1/x^2)\, y = 1/x$

=> $d(y/x)/dx = 1/x$

Integrate both sides:

$y/x = \int(1/x)\,dx = \ln|x| + c$

=> $y = x \ln|x| + c\,x$

Apply initial condition $y = -1$ when $x = 1$:

$-1 = 1\ln|1| + c1$

=> $-1 = 0 + c$

=> $c = -1$

Solution: $y = x \ln|x| - x$

(b) Solve the initial value problem $d^2y/dx^2 - 2\, dy/dx + y = xe^x$, $y(0) = -4$, $dy/dx(0) = 0$

Homogeneous equation: $d^2y/dx^2 - 2dy/dx + y = 0$

Characteristic equation: $r^2 - 2r + 1 = 0$

=> $(r - 1)^2 = 0$

=> $r = 1$ (repeated root)

Homogeneous solution: $y_h = (c_1 + c_2 x) e^x$

Particular solution: Try $y_p = ax^3 e^x$ (due to repeated root and xe^x)

$y_p = ax^3e^x$

$dy_p/dx = ax^3 e^x + 3ax^2 e^x$

$d^2y_p/dx^2 = ax^3 e^x + 3ax^2 e^x + 3ax^2 e^x + 6ax\, e^x$

Substitute into the equation:

$(ax^3 + 6ax^2 + 6ax - 2ax^3 - 6ax^2 + ax^3)e^x = (6ax)e^x = x\, e^x$

=> $6a = 1$

=> $a = 1/6$

Particular solution: $y_p = (1/6) x^3 e^x$

General solution: $y = (c1 + c2\ x) e^x + (1/6) x^3 e^x$

Apply initial conditions:

$y(0) = -4$:

$c1 = -4$

$y = (-4 + c2x)e^x + (1/6)x^3 e^x$

$dy/dx = (-4 + c2x)e^x + c2\ e^x + (1/6) x^3\ e^x + (1/2) x^2\ e^x$

$dy/dx(0) = 0$:

$-4 + c2 + 0 = 0$

$\Rightarrow c2 = 4$

Solution: $y = (-4 + 4x)e^x + (1/6) x^3e^x$

(c)(i) Temperature cools from 100°C to 80°C in 10 minutes, room temperature 25°C. Find temperature after 20 minutes.

Newton's law of cooling: $dT/dt = -k(T - T_r)$, $T_r = 25$

$dT/(T - 25) = -k\ dt$

$\int dT/(T - 25) = -k \int dt$

$\ln|T - 25| = -kt + c$

$T - 25 = e^{(-kt + c)} = Ce^{(-kt)}$

$T = 25 + Ce^{(-kt)}$

At $t = 0$, $T = 100$:

$100 = 25 + C$

$\Rightarrow C = 75$

$T = 25 + 75e^{(-kt)}$

At $t = 10$, $T = 80$:

$80 = 25 + 75e^{(-10k)}$

$\Rightarrow 55 = 75e^{(-10k)}$

=> $e^{(-10k)} = 55/75 = 11/15$

=> $-10k = \ln(11/15)$

=> $k = -\ln(11/15)/10 \approx 0.02877$

At $t = 20$:

$T = 25 + 75e^{(-20k)} = 25 + 75(11/15)^2 = 25 + 75 * 121/225 \approx 65°C$ (5 decimals: 65.00000)

(c)(ii) Time when temperature is 40°C

$40 = 25 + 75e^{(-kt)}$

$15 = 75e^{(-kt)}$

$e^{(-kt)} = 1/5$

$-kt = \ln(1/5)$

$t = -\ln(1/5)/k$

$k \approx 0.02877$

$t \approx 55.96$ minutes

(d) Solve $dy/dx = x^2y/(x + 1)$, passing through (2, 3)

$dy/y = x^2/(x + 1)\, dx$

$\int dy/y = \int x^2/(x + 1)\, dx$

$\ln|y| = \int x^2/(x + 1)\, dx$

$\int x^2/(x + 1)\, dx$:

$x^2/(x + 1) = (x^2 - 1 + 1)/(x + 1) = (x - 1) + 1/(x + 1)$

$\int[(x - 1) + 1/(x + 1)]\, dx = x^2/2 - x + \ln|x + 1| + c$

$\ln|y| = x^2/2 - x + \ln|x + 1| + c$

$y = e^{(x^2/2 - x + \ln|x + 1| + c)} = e^c |x + 1|e^{(x^2/2 - x)}$

At (2, 3):

$3 = e^c |3|e^{(4/2 - 2)} = e^c \times 3e^0$

$3 = 3e^c$

=> $e^c = 1$

$\Rightarrow y = |x + 1| e^{(x^2/2 - x)}$

8. (a)(i) Equation of tangent to ellipse $9x^2 + 25y^2 - 18x - 100y + 116 = 0$ at $(1, 5)$

Rewrite: $9(x^2 - 2x) + 25(y^2 - 4y) + 116 = 0$

Complete the square:

$9[(x - 1)^2 - 1] + 25[(y - 2)^2 - 4] + 116 = 0$

$9(x - 1)^2 + 25(y - 2)^2 - 9 - 100 + 116 = 0$

$9(x - 1)^2 + 25(y - 2)^2 = 225$

$(x - 1)^2/25 + (y - 2)^2/9 = 1$

Point $(1, 5)$ on ellipse:

$(1 - 1)^2/25 + (5 - 2)^2/9 = 0 + 9/9 = 1$ (satisfied)

Differentiate:

$2(x - 1)/25 + 2(y - 2)/9 * dy/dx = 0$

$dy/dx = -(x - 1)/25 * 9/(y - 2) = -9(x - 1)/(25(y - 2))$

At $(1, 5)$:

$dy/dx = -9(1 - 1)/(25(5 - 2)) = 0$

Tangent: $y - 5 = 0(x - 1)$

$y = 5$

(a)(ii) Normal at $P(4\cos\theta, 3\sin\theta)$ on $x^2/16 + y^2/9 = 1$ meets axes at A and B. Locus of midpoint of AB.

Point $P(4\cos\theta, 3\sin\theta)$ on ellipse (satisfies equation).

Differentiate: $x/8 + 2y/9 * dy/dx = 0$

$dy/dx = -9x/(16y)$

At P: $dy/dx = -9(4\cos\theta)/(16(3\sin\theta)) = -3\cos\theta/(4\sin\theta)$

Slope of normal $= 4\sin\theta/(3\cos\theta)$

Normal equation:

$y - 3\sin\theta = (4\sin\theta/(3\cos\theta))(x - 4\cos\theta)$

A (x-axis, $y = 0$):

$0 - 3\sin\theta = (4\sin\theta/(3\cos\theta))(x - 4\cos\theta)$

$x = \cos\theta$

$A = (\cos\theta, 0)$

B (y-axis, $x = 0$):

$y - 3\sin\theta = (4\sin\theta/(3\cos\theta))(0 - 4\cos\theta)$

$y = -\sin\theta$

$B = (0, -\sin\theta)$

Midpoint M of AB:

$((\cos\theta + 0)/2, (0 + (-\sin\theta))/2) = (\cos\theta/2, -\sin\theta/2)$

Locus: $x = \cos\theta/2$, $y = -\sin\theta/2$

$x^2 + y^2 = (\cos\theta/2)^2 + (-\sin\theta/2)^2 = 1/4$

Locus: $x^2 + y^2 = 1/4$

(b)(i) Polar equation of $x^2 + y^2 - 2x - 3y = 0$

$x = r\cos\theta$, $y = r\sin\theta$

$r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta - 3r\sin\theta = 0$

$r^2 - 2r\cos\theta - 3r\sin\theta = 0$

$r = 2\cos\theta + 3\sin\theta$

(b)(ii) Graph of $r = 2\cos\theta + 3\sin\theta$

This is a circle. Convert to Cartesian:

$r = 2\cos\theta + 3\sin\theta$

$r^2 = 2r\cos\theta + 3r\sin\theta$

$x^2 + y^2 = 2x + 3y$ (matches original)

Center: (1, 3/2), radius: sqrt(13/4)

Graph is a circle, center (1, 3/2), radius sqrt(13)/2.

(c)(i) Show line $3x - y + 1 = 0$ touches parabola $y^2 = 12x$

Solve: $y = 3x + 1$ into $y^2 = 12x$

$(3x + 1)^2 = 12x$

$9x^2 + 6x + 1 = 12x$

$9x^2 - 6x + 1 = 0$

$(3x - 1)^2 = 0$

$x = 1/3, y = 2$

Point (1/3, 2). Tangent to parabola at (1/3, 2) has same equation, confirming it touches.

(c)(ii) If $ax + by + c = 0$ touches $y^2 = -8y$, find equation connecting a, b, c

Parabola: $y^2 = -8y \Rightarrow (y + 4)^2 = 8x + 16$

Solve line with parabola, set discriminant of resulting quadratic to 0 (for tangency).

After substitution and simplification:

$2a^2 + bc = 0$