THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 ADVANCED MATHEMATICS 2

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2016**

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a)(i) Use De Moivre's theorem to find the value of $[1 + i]^4$

De Moivre's theorem: $(\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta)$

Rewrite 1 + i in polar form:

$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\arg(1 + i) = \tan^{-1}(1/1) = \pi/4$

$1 + i = \sqrt{2} [\cos(\pi/4) + i \sin(\pi/4)]$

$[1 + i]^4 = [\sqrt{2} (\cos(\pi/4) + i \sin(\pi/4))]^4$

$= (\sqrt{2})^4 [\cos(4 \pi/4) + i \sin(4 \pi/4)]$

$= 4 [\cos(\pi) + i \sin(\pi)]$

$= 4 [-1 + i0] = -4$

Answer: -4

(a)(ii) Use mathematical induction to prove $[r(\cos\theta + i \sin\theta)]^n = r^n (\cos(n\theta) + i \sin(n\theta))$

Base case (n = 1):

$[r(\cos\theta + i \sin\theta)]^1 = r^1 (\cos(1 \theta) + i \sin(1\theta))$

True.

Assume true for n = k:

$[r(\cos\theta + i \sin\theta)]^k = r^k (\cos(k\theta) + i \sin(k\theta))$

Prove for n = k + 1:

$[r(\cos\theta + i \sin\theta)]^{(k+1)} = [r(\cos\theta + i \sin\theta)]^k \times [r(\cos\theta + i \sin\theta)]$

$= [r^k (\cos(k\theta) + i \sin(k\theta))] \times [r(\cos\theta + i \sin\theta)]$

$= r^{(k+1)} [(\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta) + i (\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta)]$

$= r^{(k+1)} [\cos(k\theta + \theta) + i \sin(k\theta + \theta)]$

$= r^{(k+1)} [\cos((k+1)\theta) + i \sin((k+1)\theta)]$

True for n = k + 1. By induction, the statement holds for all positive integers n.

(b)(i) If $\arg((z - 1)/(z + 1)) = \pi/4$ and $z = x + iy$, find the locus of z in an Argand diagram

Let $z = x + iy$. Then:

$(z - 1)/(z + 1) = (x + iy - 1)/(x + iy + 1)$

$= [(x - 1) + iy]/[(x + 1) + iy]$

Rationalize: Multiply numerator and denominator by conjugate $(x + 1) - iy$

Numerator: $[(x - 1) + iy][(x + 1) - iy] = (x - 1)(x + 1) - (x - 1)iy + (x + 1)iy - iy * iy$

$= (x^2 - 1) + y^2$

Denominator: $(x + 1)^2 + y^2$

So, $(z - 1)/(z + 1) = [(x^2 + y^2 - 1) + i(2y)]/[(x + 1)^2 + y^2]$

Argument: $\tan^{-1}(2y/(x^2 + y^2 - 1)) = \pi/4$

$\Rightarrow \tan(\pi/4) = 1$

$\Rightarrow 2y/(x^2 + y^2 - 1) = 1$

$\Rightarrow 2y = x^2 + y^2 - 1$

$\Rightarrow x^2 + y^2 - 2y - 1 = 0$

$\Rightarrow x^2 + (y - 1)^2 = 2$

Locus: Circle with center (0, 1), radius sqrt(2).

(b)(ii) Solve the system $iz + w = 2i$, $z - iw = 2i$, where z and w are complex numbers

From (1): $iz + w = 2i$

From (2): $z - iw = 2i$

Multiply (2) by i:

$i(z - iw) = i \times 2i$

$\Rightarrow iz + w = -2$

Now: $iz + w = 2i$ (1)

$iz + w = -2$ (3)

This is inconsistent unless $2i = -2$, which is false.

System has no solution.

(c) One root of $z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$ is $1 + i$. Find the other roots

Given root: $1 + i$. Since coefficients are real, conjugate $1 - i$ is also a root.

Quadratic factor: $(z - (1 + i))(z - (1 - i)) = z^2 - (1 + i + 1 - i)z + (1 + i)(1 - i)$

$= z^2 - 2z + 2$

Divide $z^4 - 6z^3 + 23z^2 - 34z + 26$ by $z^2 - 2z + 2$:

$z^4 - 6z^3 + 23z^2 - 34z + 26 = (z^2 - 2z + 2)(z^2 - 4z + 13)$

Solve $z^2 - 4z + 13 = 0$:

Discriminant: $(-4)^2 - 4 * 13 = 16 - 52 = -36$

$z = (4 \pm \sqrt{-36})/2 = 2 \pm 3i$

Roots: $1 + i$, $1 - i$, $2 + 3i$, $2 - 3i$.

2. (a)(i) Draw the simplified electrical circuit for $[(p \wedge (p \vee q))] \wedge [q \to (p \wedge q)]$

Simplify $[(p \wedge (p \vee q))] \wedge [q \to (p \wedge q)]$:

$p \wedge (p \vee q) = p$ (absorption law)

$q \to (p \wedge q) = \neg q \vee (p \wedge q)$

$= (\neg q \vee p) \wedge (\neg q \vee q) = \neg q \vee p$

So, expression $= p \wedge (\neg q \vee p) = p$

Circuit: Single wire (p).

(a)(ii) Use truth table to show whether or not $p \to q$ and $\neg(p \wedge q)$ are logically equivalent

P	q	p --> q	p ∧ q	¬(p ∧ q)
T	T	T	T	F
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$p \to q \neq \neg(p \wedge q)$. Not logically equivalent.

(b)(i) Use truth table to test validity: If I am intelligent, then I will pass this examination. I am intelligent. Therefore I will pass this examination.

Let p: I am intelligent, q: I will pass this examination.

Premises: p --> q, p. Conclusion: q.

P	q	p --> q	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

When p --> q and p are true, q is true. Argument is valid.

(b)(ii) Write converse, inverse, and contrapositive of "If Mathematics is interesting then Biology is boring and tough"

Statement: p --> (q ∧ r), p: Mathematics is interesting, q: Biology is boring, r: Biology is tough.

Converse: (q ∧ r) --> p

If Biology is boring and tough, then Mathematics is interesting.

Inverse: ¬p --> ¬(q ∧ r)

If Mathematics is not interesting, then Biology is not boring or not tough.

Contrapositive: ¬(q ∧ r) --> ¬p

If Biology is not boring or not tough, then Mathematics is not interesting.

3. (a)(i) If position vectors OA, OB, OC are OA = 2i - j + 3k, OB = 3i + 2j - 4k, OC = -i + 3j - 2k, determine AB x BC

AB = OB - OA = (3i + 2j - 4k) - (2i - j + 3k) = i + 3j - 7k

BC = OC - OB = (-i + 3j - 2k) - (3i + 2j - 4k) = -4i + j + 2k

AB x BC = | i j k |

| 1 3 -7 |

| -4 1 2 |

= i(32 - (-7)1) - j(12 - (-7)(-4)) + k(11 - 3*(-4))

= i(6 + 7) - j(2 - 28) + k(1 + 12)

$= 13i + 26j + 13k$

(a)(ii) Find the exact value of the angle between AB and BC

$|AB| = \text{sqrt}(1^2 + 3^2 + (-7)^2) = \text{sqrt}(59)$

$|BC| = \text{sqrt}((-4)^2 + 1^2 + 2^2) = \text{sqrt}(21)$

$AB . BC = 1(-4) + 3x1 + (-7)*2 = -4 + 3 - 14 = -15$

$\cos\theta = (AB . BC)/(|AB||BC|) = -15/(\text{sqrt}(59) \times \text{sqrt}(21))$

Exact value: $-15/\text{sqrt}(1239)$

(b)(i) If $a = 3i + 2j - 9k$, $b = i + j + 3k$, find value of k so that $a + k b$ is perpendicular to $a - b$

$a + k b = (3 + k)i + (2 + k)j + (-9 + 3k)k$

$a - b = (3 - 1)i + (2 - 1)j + (-9 - 3)k = 2i + j - 12k$

Dot product = 0:

$(3 + k)2 + (2 + k)1 + (-9 + 3k)(-12) = 0$

$6 + 2k + 2 + k + 108 - 36k = 0$

$-33k + 116 = 0$

$k = 116/33$

(b)(ii) Projection of a onto b in surd form

$|a| = \text{sqrt}(3^2 + 2^2 + (-9)^2) = \text{sqrt}(94)$

$|b| = \text{sqrt}(1^2 + 1^2 + 3^2) = \text{sqrt}(11)$

$a . b = 31 + 21 + (-9)3 = 3 + 2 - 27 = -22$

Projection of a onto b = $(a . b)/|b| = -22/\text{sqrt}(11)$

(c) Derive the cosine rule using vectors a and b

Let $c = a - b$.

$c^2 = (a - b) . (a - b) = a^2 + b^2 - 2a . b$

$a^2 + b^2 - c^2 = 2a . b$

$a . b = |a||b|\cos\theta$

$\cos\theta = (a^2 + b^2 - c^2)/(2|a||b|)$

Cosine rule: $\cos\theta = (a^2 + b^2 - c^2)/(2ab)$

4. (a)(i) Find the value of a if the 17th and 18th terms of the expansion $(2 + a)^n$ are equal

For $(2 + a)^n$, the general term is:

$T(r+1) = C(n,r) \times 2^{(n-r)} \times a^r$

17th term (r = 16): $T(17) = C(n,16) \times 2^{(n-16)} \times a^{16}$

18th term (r = 17): $T(18) = C(n,17) \times 2^{(n-17)} \times a^{17}$

Given $T(17) = T(18)$:

$C(n,16) \times 2^{(n-16)} \times a^{16} = C(n,17) \times 2^{(n-17)} \times a^{17}$

$C(n,16)/C(n,17) \times 2^{(n-16-(n-17))} = a^{(17-16)}$

$C(n,16)/C(n,17) = (n-17+1)/(17) = (n-16)/17$

So, $(n-16)/17 \times 2^1 = a$

=> $a = 2(n-16)/17$

We need n (not given). Assuming binomial context, if n is an integer, a depends on n. Without n, we leave:

$a = 2(n-16)/17$

(a)(ii) The roots of $x^3 + px^2 + qx + 30 = 0$ are in the ratio 2:3:5. Find p and q

Let roots be 2k, 3k, 5k.

Sum of roots: $2k + 3k + 5k = 10k = -p$

Product of roots: $2k \times 3k \times 5k = 30k^3 = -30$

Sum of pairwise products: $(2k)(3k) + (2k)(5k) + (3k)(5k) = 6k^2 + 10k^2 + 15k^2 = 31k^2 = q$

From product: $30k^3 = -30$

=> $k^3 = -1$

=> $k = -1$

Sum of roots: $10k = -10 = -p$

=> $p = 10$

Pairwise products: $31k^2 = 31(-1)^2 = 31 = q$

$\Rightarrow q = 31$

Answer: $p = 10$, $q = 31$

(b)(i) State the principle of mathematical induction as it is used in mathematics

Mathematical induction: To prove a statement $P(n)$ for all positive integers n:

Base case: Prove $P(1)$ is true.

Inductive step: Assume $P(k)$ is true for some $k \geq 1$ (inductive hypothesis), then prove $P(k+1)$ is true.

(b)(ii) Use mathematical induction to prove $\Sigma(3r-1) = n/2(3n + 1)$, r from 1 to n

Base case ($n = 1$):

Left: $\Sigma(3r-1) = 3(1)-1 = 2$

Right: $1/2 \times (3(1) + 1) = 1/2 \times 4 = 2$

True.

Assume true for $n = k$:

$\Sigma(3r-1) = k/2 \times (3k + 1)$

Prove for $n = k + 1$:

$\Sigma(3r-1)$ from $r=1$ to $k+1$ = [$\Sigma(3r-1)$ from $r=1$ to k] + $(3(k+1)-1)$

$= k/2 \times (3k + 1) + (3k + 2)$

$= (3k^2 + k)/2 + (3k + 2)$

$= (3k^2 + k + 6k + 4)/2$

$= (3k^2 + 7k + 4)/2$

$= (k+1)/2 \times (3k + 4)$

$= (k+1)/2 \times (3(k+1) + 1)$

Matches the form. True for $n = k + 1$. By induction, the statement holds for all positive integers n.

5. (a)(i) Solve $\tan^{-1}((x-1)/(x-2)) + \tan^{-1}((x+1)/(x+2)) = \pi/4$, leave answer in surd form

Let $A = \tan^{-1}((x-1)/(x-2))$, $B = \tan^{-1}((x+1)/(x+2))$.

Given: $A + B = \pi/4$

$\Rightarrow \tan(A + B) = \tan(\pi/4) = 1$

Using $\tan(A + B) = (\tan A + \tan B)/(1 - \tan A \tan B)$:

$\tan A = (x-1)/(x-2)$, $\tan B = (x+1)/(x+2)$

Numerator: $(x-1)/(x-2) + (x+1)/(x+2) = [(x-1)(x+2) + (x+1)(x-2)]/[(x-2)(x+2)]$

$= (x^2 + x - 2 + x^2 - x - 2)/(x^2 - 4) = (2x^2 - 4)/(x^2 - 4)$

Denominator: $1 - (x-1)/(x-2) \text{ x } (x+1)/(x+2) = 1 - [(x-1)(x+1)]/[(x-2)(x+2)]$

$= 1 - (x^2 - 1)/(x^2 - 4) = (x^2 - 4 - x^2 + 1)/(x^2 - 4) = (-3)/(x^2 - 4)$

So, $\tan(A + B) = [(2x^2 - 4)/(x^2 - 4)] / [(-3)/(x^2 - 4)] = (2x^2 - 4)/(-3) = 1$

$\Rightarrow 2x^2 - 4 = -3$

$\Rightarrow 2x^2 = 1$

$\Rightarrow x^2 = 1/2$

$\Rightarrow x = \pm\sqrt{(1/2)} = \pm\sqrt{2}/2$

Answer: $x = \sqrt{2}/2$ or $x = -\sqrt{2}/2$

(a)(ii) Prove $1 + \sin x/(1 - \sin x) = (\tan x + \sec x)^2$

Right side: $(\tan x + \sec x)^2 = \tan^2 x + 2\tan x \sec x + \sec^2 x$

$= (\sin x/\cos x)^2 + 2(\sin x/\cos x)(1/\cos x) + (1/\cos x)^2$

$= (\sin^2 x + 2\sin x + 1)/\cos^2 x = (1 + \sin x)^2/(1 - \sin^2 x) = (1 + \sin x)^2/((1 - \sin x)(1 + \sin x))$

$= (1 + \sin x)/(1 - \sin x)$

Left side matches right side. Identity proven.

(b) If $2\sin\theta - \cos\theta = 1$, use t-formula to find θ in $0° \le \theta \le 180°$

Let $t = \tan(\theta/2)$, then:

$\sin\theta = 2t/(1 + t^2)$, $\cos\theta = (1 - t^2)/(1 + t^2)$

Substitute: $2(2t/(1 + t^2)) - (1 - t^2)/(1 + t^2) = 1$

$\Rightarrow (4t - (1 - t^2))/(1 + t^2) = 1$

$\Rightarrow 4t - 1 + t^2 = 1 + t^2$

$\Rightarrow 4t = 2$

$\Rightarrow t = 1/2$

Since $t = \tan(\theta/2)$, $\tan(\theta/2) = 1/2$

$\theta/2 = \tan^{-1}(1/2) \approx 26.57°$, so $\theta \approx 53.14°$

In $0° \leq \theta \leq 180°$, $\theta = 53.14°$.

(c)(i) Show $\cos\theta + \cos2\theta + \cos3\theta + \cos4\theta = \cos(5\theta/2)\cos(3\theta/2)/\sin(\theta/2)$

Left: $\cos\theta + \cos2\theta + \cos3\theta + \cos4\theta$

Use sum-to-product: $\cos A + \cos B = 2\cos((A+B)/2)\cos((A-B)/2)$

Pair: $(\cos\theta + \cos4\theta) + (\cos2\theta + \cos3\theta)$

$\cos\theta + \cos4\theta = 2\cos(5\theta/2)\cos(-3\theta/2) = 2\cos(5\theta/2)\cos(3\theta/2)$

$\cos2\theta + \cos3\theta = 2\cos(5\theta/2)\cos(\theta/2)$

Total: $2\cos(5\theta/2)\cos(3\theta/2) + 2\cos(5\theta/2)\cos(\theta/2) = 2\cos(5\theta/2)[\cos(3\theta/2) + \cos(\theta/2)]$

$\cos(3\theta/2) + \cos(\theta/2) = 2\cos(\theta)\cos(\theta/2)$

So, left $= 4\cos(5\theta/2)\cos(\theta)\cos(\theta/2)$

Right: $\cos(5\theta/2)\cos(3\theta/2)/\sin(\theta/2)$

Use $\cos A \cos B = (1/2)[\cos(A+B) + \cos(A-B)]$:

$\cos(5\theta/2)\cos(3\theta/2) = (1/2)[\cos(4\theta) + \cos(\theta)]$

Right $= (1/2)[\cos(4\theta) + \cos(\theta)]/\sin(\theta/2)$

Right $= \cos(5\theta/2)\cos(3\theta/2)/\sin(\theta/2)$

(c)(ii) Verify $\sin(A + B + C) + \sin(A - B - C) - \cos(A + B - C) - \cos(A - B + C) = \tan B \tan C$

This identity appears incorrect as stated. Test numerically (e.g., $A = 30°$, $B = 45°$, $C = 60°$):

Left $\neq \tan B \tan C$. Likely a typo in the problem. If intended identity is different, please clarify.

(d) Express $3\sin\theta - 4\cos\theta$ in the form $R\sin(\theta - \alpha)$, giving R and α

$3\sin\theta - 4\cos\theta = R\sin(\theta - \alpha) = R(\sin\theta \cos\alpha - \cos\theta \sin\alpha)$

Equate: $R \cos\alpha = 3$, $-R \sin\alpha = -4$

=> $R \sin\alpha = 4$

$R^2 = 3^2 + 4^2 = 25$, $R = 5$

$\cos\alpha = 3/5$, $\sin\alpha = 4/5$

$\alpha = \tan^{-1}(4/3)$

Answer: $5\sin(\theta - \tan^{-1}(4/3))$

6. (a)(i) Probability a keyboard is defective is 0.01. For a sample of 3, construct the probability distribution of defective keyboards

Let X = number of defective keyboards. X ~ Binomial(n=3, p=0.01).

$P(X = 0) = C(3,0) \times (0.01)^0 \times (0.99)^3 = 0.970299$

$P(X = 1) = C(3,1) \times (0.01)^1 \times (0.99)^2 = 0.029403$

$P(X = 2) = C(3,2) \times (0.01)^2 \times (0.99)^1 = 0.000297$

$P(X = 3) = C(3,3) \times (0.01)^3 \times (0.99)^0 = 0.000001$

Distribution:

X	0	1	2	3
P(X)	0.970299	0.029403	0.000297	0.000001

(a)(ii) Find the mean and standard deviation (to 2 decimal places)

Mean: $E(X) = n \times p = 3 \times 0.01 = 0.03$

Variance: $Var(X) = n \times p \times (1 - p) = 3 \times 0.01 \times 0.99 = 0.0297$

Standard deviation: $\sqrt{(0.0297)} \approx 0.17$

Answer: Mean = 0.03, SD = 0.17

(b) Bag R: 5 red, 3 green; Bag P: 7 red, 5 green. Draw 2 from R, 1 from P. Find probability 2 red, 1 green

Total ways: $C(8,2) \times C(12,1) = 28 \times 12 = 336$

Favorable: 2 red from R = $C(5,2) = 10$, 1 green from P = 5

Probability = $(10 \times 5)/336 = 50/336 = 25/168$

(c)(i) $P(X = k) = k$ if $x = 0$, $2k$ if $x = 1$, $3k$ if $x = 2$, 0 otherwise. Determine k

Sum of probabilities = 1:

$k + 2k + 3k = 1$

$6k = 1$

$k = 1/6$

(c)(ii) Find $P(X < 2)$, $P(X \leq 2)$, $P(X \geq 2)$

$P(X = 0) = 1/6$, $P(X = 1) = 2/6 = 1/3$, $P(X = 2) = 3/6 = 1/2$

$P(X < 2) = P(X = 0) + P(X = 1) = 1/6 + 1/3 = 1/2$

$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 1$

$P(X \geq 2) = P(X = 2) = 1/2$

(d) If X is discrete, show $E(aX + b) = aE(X) + b$

$E(aX + b) = \Sigma\ (ax + b) \times P(X = x)$

$= a\Sigma\ x \times P(X = x) + b\Sigma\ P(X = x)$

$= aE(X) + b \times 1$

$= aE(X) + b$

(e) 6 seeds, probability of germinating 0.9. Probability at most 5 seeds germinate

$X \sim \text{Binomial}(n=6, p=0.9)$.

P(at most 5) = $1 - P(X = 6)$

$P(X = 6) = (0.9)^6 = 0.531441$

$P(X \leq 5) = 1 - 0.531441 = 0.468559 \approx 0.47$

7. (a)(i) If $x(1 - y)\ dy/dx - 2y = 0$ and $y = 2$ when $x = e$, show that $x^2y\ e^y = e^2$

Rewrite: $x(1 - y)\ dy/dx - 2y = 0$

$\Rightarrow x(1 - y)\ dy/dx = 2y$

$\Rightarrow (1 - y)\ dy/dx = 2y/x$

$\Rightarrow dy/dx = 2y/[x(1 - y)]$

Separate variables:

$(1 - y)/(2y)\ dy = 1/x\ dx$

$\Rightarrow (1/(2y) - 1/2)\,dy = 1/x\,dx$

Integrate:

$(1/2)\ln|y| - (1/2)y = \ln|x| + c$

Apply condition $y = 2$ when $x = e$:

$(1/2)\ln|2| - (1/2)(2) = \ln|e| + c$

$\Rightarrow (1/2)\ln 2 - 1 = 1 + c$

$\Rightarrow c = (1/2)\ln 2 - 2$

Equation: $(1/2)\ln|y| - (1/2)y = \ln|x| + (1/2)\ln 2 - 2$

$\Rightarrow \ln|y| - y = 2\ln|x| + \ln 2 - 4$

$\Rightarrow \ln|y| - y = \ln(x^2) + \ln 2 - 4$

$\Rightarrow \ln(y/x^2) - y = \ln 2 - 4$

$\Rightarrow y/x^2 = e^{(y + \ln 2 - 4)} = (2/e^4)\,e^y$

$\Rightarrow x^2y\,e^y = e^4/2 \times 2 = e^4$

Show: $x^2y\,e^y = e^2$

From above, $x^2y\,e^y = e^4$, which does not match e^2. Likely a typo in the problem (should be e^4).

(a)(ii) Solve $2x - y\,dy/dx = 2x + y + 2$ given $y = 1$ when $x = 2$

Rewrite: $2x - y\,dy/dx = 2x + y + 2$

$\Rightarrow y\,dy/dx = -2$

Integrate:

$y^2/2 = -2x + c$

$\Rightarrow y^2 = -4x + c$

At $x = 2$, $y = 1$:

$1 = -4(2) + c$

$\Rightarrow c = 9$

$\Rightarrow y^2 = -4x + 9$

$\Rightarrow y = \pm\sqrt{(-4x + 9)}$

Since $y = 1$ at $x = 2$, $y = \sqrt{-4x + 9}$ (positive branch).

(b) Form a differential equation whose solution is $y = A e^{2x} + B e^{-x}$ where A and B are arbitrary constants

$y = A e^{2x} + B e^{-x}$

$dy/dx = 2A e^{2x} - B e^{-x}$

$d^2y/dx^2 = 4A e^{2x} + B e^{-x}$

Express in terms of y:

$d^2y/dx^2 = 4A e^{2x} + B e^{-x}$

$y = A e^{2x} + B e^{-x}$

$dy/dx - 2y = (2A e^{2x} - B e^{-x}) - 2(A e^{2x} + B e^{-x}) = -3B e^{-x}$

$d^2y/dx^2 - 2\, dy/dx = 4A e^{2x} + B e^{-x} - 2(2A e^{2x} - B e^{-x}) = -2y$

Differential equation: $d^2y/dx^2 - 2\, dy/dx + 2y = 0$

(c)(i) Tank with 1000 liters, 10 kg salt. Poured off at 20 liters/min, pure water added at 20 liters/min. Find mass of salt after 5 minutes

Let S(t) be salt mass at time t (in kg). Volume remains 1000 liters.

Salt concentration = S/1000 kg/liter

Salt poured off: $(S/1000) \times 20 = S/50$ kg/min

$dS/dt = -S/50$

$dS/S = -dt/50$

$\ln|S| = -t/50 + c$

$S = c\, e^{(-t/50)}$

At $t = 0$, $S = 10$:

$10 = c$

$S = 10\, e^{(-t/50)}$

At $t = 5$:

$S = 10 e^{(-5/50)} = 10 e^{(-0.1)} \approx 9.05$ kg

(c)(ii) How long until mass of salt falls to 5 kg?

$5 = 10 e^{(-t/50)}$

$e^{(-t/50)} = 1/2$

$-t/50 = \ln(1/2)$

$t = 50 \ln(2) \approx 34.66$ minutes

(d) Find the general solution of $dy/dx - 4\, dy/dy + 3y = 10 e^{3x}$

Corrected: $d^2y/dx^2 - 4\, dy/dx + 3y = 10 e^{3x}$

Homogeneous: $d^2y/dx^2 - 4\, dy/dx + 3y = 0$

Characteristic: $r^2 - 4r + 3 = 0$

$\Rightarrow (r - 1)(r - 3) = 0$

$\Rightarrow r = 1, 3$

$y_h = c1\, e^x + c2\, e^{3x}$

Particular: $y_p = A e^{3x}$

$y_p' = 3A e^{3x}$, $y_p'' = 9A e^{3x}$

$9A e^{3x} - 4(3A e^{3x}) + 3(A e^{3x}) = 10 e^{3x}$

$y_p = Ax e^{3x}$:

$y_p' = A e^{3x} + 3Ax e^{3x}$, $y_p'' = 6A e^{3x} + 9Ax e^{3x}$

$(6A + 9Ax - 12A - 12Ax + 3Ax) e^{3x} = 10 e^{3x}$

$(6A - 3Ax) e^{3x} = 10 e^{3x}$

$6A = 10$, $A = 5/3$

General solution: $y = c1\, e^x + c2\, e^{3x} + (5/3)x e^{3x}$

8. (a)(i) If $9y^2 - 54y - 25x^2 + 200x - 544 = 0$ is the hyperbola equation, find center, vertices, foci, equation of asymptotes

Rewrite: $9(y^2 - 6y) - 25(x^2 - 8x) - 544 = 0$

$9[(y - 3)^2 - 9] - 25[(x - 4)^2 - 16] = 544$

$9(y - 3)^2 - 25(x - 4)^2 = 225$

$(y - 3)^2/25 - (x - 4)^2/9 = 1$

Center: (4, 3)

$a = 5$, $b = 3$

Vertices: $(4, 3\pm5) = (4, 8), (4, -2)$

$c = \sqrt{(a^2 + b^2)} = \sqrt{(25 + 9)} = \sqrt{34}$

Foci: $(4, 3\pm\sqrt{34})$

Asymptotes: $(y - 3) = \pm(5/3)(x - 4)$

(a)(ii) Convert $x + y = 4x$ into polar equation

$x = r \cos\theta$, $y = r \sin\theta$

$r \cos\theta + r \sin\theta = 4(r \cos\theta)$

$r(\cos\theta + \sin\theta - 4\cos\theta) = 0$

$r = 0$ or $\cos\theta + \sin\theta - 4\cos\theta = 0$

$\Rightarrow r = 0$ or $r = 3\cos\theta/(\cos\theta + \sin\theta)$

Polar equation: $r = 3\cos\theta/(\cos\theta + \sin\theta)$

(a)(iii) Convert [1, -1] into polar coordinates

Cartesian (1, -1):

$r = \sqrt{(1^2 + (-1)^2)} = \sqrt{2}$

$\theta = \tan^{-1}(-1/1) = -\pi/4$

Polar: $(\sqrt{2}, -\pi/4)$

(b) Find equation of tangent and normal at $P(a\cos\alpha, b\sin\alpha)$ to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$

Ellipse: $x^2/a^2 + y^2/b^2 = 1$

Point P($a\cos\alpha$, $b\sin\alpha$).

Differentiate: $2x/a^2 + 2y/b^2\ dy/dx = 0$

$dy/dx = -(b^2x)/(a^2y)$

At P: $dy/dx = -(b^2 a\cos\alpha)/(a^2 b\sin\alpha) = -(b/a)\cot\alpha$

Tangent: $y - b\sin\alpha = -(b/a)\cot\alpha\ (x - a\cos\alpha)$

=> $bx \cot\alpha + ay = ab(\cot\alpha \cos\alpha + \sin\alpha)$

Normal slope: $(a/b)\tan\alpha$

$y - b\sin\alpha = (a/b)\tan\alpha\ (x - a\cos\alpha)$

(c)(i) Define a conic section

A conic section is the curve obtained by intersecting a plane with a double cone. It can be a circle, ellipse, parabola, or hyperbola, depending on the angle of intersection.

(c)(ii) Man running a race-course, sum of distances from two flag posts to him is always 10 meters. Sum of distance between flag posts is 8 meters. Find equation of path traced by the man

This is an ellipse: sum of distances to foci = $2a = 10$, $a = 5$.

Distance between foci = $2ae = 8$, $ae = 4$

$e = 4/5$

$b^2 = a^2(1 - e^2) = 25(1 - (4/5)^2) = 25(1 - 16/25) = 9$

Foci at $(\pm4, 0)$, center at $(0, 0)$.

Equation: $x^2/25 + y^2/9 = 1$

(d) Sketch the graph of $r = 2(1 + \sin\theta)$

This is a cardioid.

At $\theta = 0$, $r = 2$; $\theta = \pi/2$, $r = 4$; $\theta = \pi$, $r = 2$; $\theta = 3\pi/2$, $r = 0$.

Graph: Heart-shaped curve, cusp at (0, 0), extends to (0, 4).