THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 **ADVANCED MATHEMATICS 2**

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2021**

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

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1. (a) If A and B are such that $P(A) = 1/3$, $P(B) = 1/4$, and $P(A \cup B) = 1/2$, calculate:

(i) $P(A \cap B')$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$1/2 = 1/3 + 1/4 - P(A \cap B)$

$P(A \cap B) = 1/3 + 1/4 - 1/2 = 4/12 + 3/12 - 6/12 = 1/12$

$P(A \cap B') = P(A) - P(A \cap B) = 1/3 - 1/12 = 4/12 - 1/12 = 3/12 = 1/4$

Answer: $P(A \cap B') = 1/4$.

(ii) $P(A' | B')$

$P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - 1/2 = 1/2$

$P(B') = 1 - P(B) = 1 - 1/4 = 3/4$

$P(A' | B') = P(A' \cap B') / P(B') = (1/2) / (3/4) = 1/2 \times 4/3 = 2/3$

Answer: $P(A' | B') = 2/3$.

1. (b) Two dice are thrown simultaneously.

(i) List the sample space for this event.

Sample space (pairs (a, b) where a, b are numbers on each die, 1 to 6):

(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),

(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),

(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),

(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),

(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),

(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)

Total outcomes = $6 \times 6 = 36$

Answer: Sample space listed as above (36 outcomes).

(ii) Find the probability that the sum of the numbers obtained on the dice is neither a multiple of 2 nor a multiple of 3.

Possible sums: 2 to 12

Multiples of 2: 2, 4, 6, 8, 10, 12

Multiples of 3: 3, 6, 9, 12

Multiples of 2 or 3: 2, 3, 4, 6, 8, 9, 10, 12

Count outcomes for each sum:

Sum 2: (1,1) → 1

Sum 3: (1,2), (2,1) → 2

Sum 4: (1,3), (2,2), (3,1) → 3

Sum 6: (1,5), (2,4), (3,3), (4,2), (5,1) → 5

Sum 8: (2,6), (3,5), (4,4), (5,3), (6,2) → 5

Sum 9: (3,6), (4,5), (5,4), (6,3) → 4

Sum 10: (4,6), (5,5), (6,4) → 3

Sum 12: (6,6) → 1

Total outcomes for multiples of 2 or 3 = 1 + 2 + 3 + 5 + 5 + 4 + 3 + 1 = 24

Outcomes neither multiple of 2 nor 3 = 36 - 24 = 12

Probability = 12 / 36 = 1/3

Answer: Probability = 1/3.

1. (c) If X is binomially distributed, the probability that the event will happen exactly x times in n trials is given by the function $P(X = x) = (n \text{ choose } x)\, p^x (1-p)^{(n-x)}$. Establish the validity of the Poisson approximation to the binomial distribution.

For binomial: mean = np, variance = np(1-p).

Poisson approximation applies when n is large, p is small, $\lambda = np$ is moderate.

$P(X = x) = (n! / (x! (n-x)!))\, p^x (1-p)^{(n-x)}$

Substitute $p = \lambda/n$:

$= (n (n-1) \ldots (n-x+1) / x!)\, (\lambda/n)^x (1 - \lambda/n)^{(n-x)}$

As $n \to \infty$:

$n (n-1) \ldots (n-x+1) / n^x \to 1$

$(1 - \lambda/n)^{(n-x)} \to e^{(-\lambda)}$

Result: $(\lambda^x / x!) e^{(-\lambda)}$, which is the Poisson probability.

Answer: Poisson approximation is valid for large n, small p, with $\lambda = np$.

2. (a) The contrapositive of the statement Y is given by $\neg(Q \land P) \to \neg P$. By using the laws of algebra of propositions, show that its inverse is a tautology.

First, identify the original statement Y. The contrapositive $\neg(Q \land P) \to \neg P$ implies the original statement Y is $P \to (Q \land P)$.

Original: $P \to (Q \land P)$

Contrapositive: $\neg(Q \land P) \to \neg P$ (which matches the given statement).

Now, find the inverse of the original statement Y ($P \to Q \land P$):

Inverse of $P \to (Q \land P)$ is $\neg P \to \neg(Q \land P)$.

Simplify: $\neg P \to (\neg Q \lor \neg P)$ (using De Morgan's law).

In implication form: $\neg P \to (\neg Q \lor \neg P) = \neg\neg P \lor (\neg Q \lor \neg P) = P \lor \neg Q \lor \neg P = (P \lor \neg P) \lor \neg Q = 1 \lor \neg Q = 1$ (since $P \lor \neg P$ is always true).

The inverse simplifies to a tautology (always true).

Answer: The inverse of the statement is a tautology, as shown: $\neg P \to (\neg Q \lor \neg P)$ simplifies to 1.

2. (c) (i) Construct a truth table for the compound statement that corresponds to the following circuit:

Circuit: P and R in parallel, both in series with Q.

Statement: $Q \land (P \lor R)$

Truth table:

P	R	Q	P ∨ R	Q ∧ (P ∨ R)
0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0

1 | 0 | 1 | 1 | 1

1 | 1 | 0 | 1 | 0

1 | 1 | 1 | 1 | 1

Answer: Truth table as above.

2. (c) (ii) Draw a simple network diagram for the statement $(P \to Q) \wedge (P \vee Q)$.

$(P \to Q) \wedge (P \vee Q) = (\neg P \vee Q) \wedge (P \vee Q)$

Network: $(\neg P \vee Q)$ and $(P \vee Q)$ in series.

$\neg P$ and Q in parallel, in series with P and Q in parallel.

Answer: Network: $(\neg P \vee Q)$ and $(P \vee Q)$ in series.

3. (a) If a = i - j + 2k and b = i + j, find the unit vector orthogonal to both a and b.

Cross product $a \times b$:

$i ((-1) \times 0 - 2 \times 1) - j (1 \times 0 - 2 \times 1) + k (1 \times 1 - (-1) \times 1)$

$= i (-2) - j (-2) + k (1 + 1) = -2i + 2j + 2k$

Magnitude: $\sqrt{(-2)^2 + 2^2 + 2^2} = \sqrt{4 + 4 + 4} = \sqrt{12} = 2\sqrt{3}$

Unit vector: $(-2i + 2j + 2k) / (2\sqrt{3}) = (-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

Answer: Unit vector = $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

3. (b) The position vectors of the points A and B are 2i - 3j + k and -i + 5j + 7k respectively. If C divides AB internally in the ratio 2:1, find the position vector of point C.

C divides AB in 2:1:

$C = (1 \times A + 2 \times B) / (1 + 2)$

$= (1 (2i - 3j + k) + 2 (-i + 5j + 7k)) / 3$

$= (2i - 3j + k - 2i + 10j + 14k) / 3$

$= (7j + 15k) / 3$

$= (7/3) j + 5 k$

Answer: Position vector of C = $(7/3) j + 5 k$.

3. (c) Using the cosine rule, show that in the triangle ABC, c = b cos A + a cos B.

Cosine rule:

$c^2 = a^2 + b^2 - 2ab \cos C$

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

From $a^2 = b^2 + c^2 - 2bc \cos A$:

$2bc \cos A = b^2 + c^2 - a^2$

From $b^2 = a^2 + c^2 - 2ac \cos B$:

$2ac \cos B = a^2 + c^2 - b^2$

Add: $2bc \cos A + 2ac \cos B = (b^2 + c^2 - a^2) + (a^2 + c^2 - b^2) = 2c^2$

$bc \cos A + ac \cos B = c^2$

$c = b \cos A + a \cos B$

Answer: Proven, $c = b \cos A + a \cos B$.

5. (a) If A, B, and C are angles of a right-angled triangle such that $\cos A = 3/5$ and $\cos B = 5/13$, find the value of $\tan 2A$, $\cos (A + B)$, and $\text{cosec} (A - B)$ in the form x/y.

Since A, B, and C are angles in a right-angled triangle, $A + B + C = 180°$, and $C = 90°$ (right angle), so $A + B = 90°$, $B = 90° - A$.

Given $\cos A = 3/5$:

$\sin A = \sqrt{(1 - (3/5)^2)} = \sqrt{(1 - 9/25)} = \sqrt{(16/25)} = 4/5$

Given $\cos B = 5/13$, and $B = 90° - A$, so $\cos B = \sin A = 4/5$ (matches given 5/13, but let's proceed and check):

$\sin B = \cos A = 3/5$

$\tan 2A$: $\tan 2A = 2 \tan A / (1 - \tan^2 A)$, $\tan A = \sin A / \cos A = (4/5) / (3/5) = 4/3$

$\tan 2A = 2 (4/3) / (1 - (4/3)^2) = (8/3) / (1 - 16/9) = (8/3) / (-7/9) = (8/3) \times (-9/7) = -24/7$

$\cos (A + B) = \cos (90°) = 0$

$\text{cosec} (A - B)$: $A - B = A - (90° - A) = 2A - 90°$, $\text{cosec} (2A - 90°) = \sec 2A$

$\sec 2A = 1 / \cos 2A$, $\cos 2A = 1 - 2 \sin^2 A = 1 - 2 (4/5)^2 = 1 - 2 (16/25) = 1 - 32/25 = -7/25$

$\sec 2A = -25/7$, $\operatorname{cosec}(A - B) = -25/7$

Answer: $\tan 2A = -24/7$, $\cos(A + B) = 0$, $\operatorname{cosec}(A - B) = -25/7$.

5. (b) (i) Show that $\cot(x + \pi/2) - \tan(x - \pi/2) = 2 \cos 2x / \sin 2x$.

$\cot(x + \pi/2) = -\tan x$

$\tan(x - \pi/2) = -\cot x$

$\cot(x + \pi/2) - \tan(x - \pi/2) = -\tan x - (-\cot x) = -\tan x + \cot x$

$= -(\sin x / \cos x) + (\cos x / \sin x) = (\cos^2 x - \sin^2 x) / (\sin x \cos x)$

$= (\cos 2x) / ((1/2) \sin 2x) = 2 \cos 2x / \sin 2x$

Answer: Proven.

(ii) Solve the equation $4 \cos 2\theta - 2 \cos \theta + 3 = 0$, for $0° \leq \theta \leq 360°$.

$4 \cos 2\theta - 2 \cos \theta + 3 = 0$

Use $\cos 2\theta = 2 \cos^2 \theta - 1$:

$4(2 \cos^2 \theta - 1) - 2 \cos \theta + 3 = 0$

$8 \cos^2 \theta - 4 - 2 \cos \theta + 3 = 0$

$8 \cos^2 \theta - 2 \cos \theta - 1 = 0$

Let $u = \cos \theta$:

$8u^2 - 2u - 1 = 0$

$u = (2 \pm \sqrt{(4 + 32)}) / 16 = (2 \pm 6) / 16$

$u = 1/2$ or $u = -1/4$

$\cos \theta = 1/2$: $\theta = 60°, 300°$

$\cos \theta = -1/4$: $\theta = 104.5°, 255.5°$ (approximate, within range)

Answer: $\theta = 60°, 104.5°, 255.5°, 300°$.

5. (c) Express $\cos^4 \theta$ in terms of cosines multiples of θ.

$\cos^4 \theta = (\cos^2 \theta)^2 = ((1 + \cos 2\theta) / 2)^2 = (1 + 2 \cos 2\theta + \cos^2 2\theta) / 4$

$\cos^2 2\theta = (1 + \cos 4\theta) / 2$

$\cos^4 \theta = (1 + 2 \cos 2\theta + (1 + \cos 4\theta) / 2) / 4 = (2 + 4 \cos 2\theta + 1 + \cos 4\theta) / 8$

$= (3 + 4 \cos 2\theta + \cos 4\theta) / 8$

Answer: $\cos^4 \theta = (3 + 4 \cos 2\theta + \cos 4\theta) / 8$.

6. (a) By using the first five terms in the expansion of $(1 + x)^n$, find the value of $(1.98)^{10}$ correct to three decimal places.

$(1.98)^{10} = (2 - 0.02)^{10} = 2^{10} (1 - 0.01)^{10}$

$(1 - 0.01)^{10} \approx 1 + 10 (-0.01) + (10 \times 9 / 2) (-0.01)^2 + (10 \times 9 \times 8 / 6) (-0.01)^3 + (10 \times 9 \times 8 \times 7 / 24) (-0.01)^4$

$= 1 - 0.1 + 45 (0.0001) + 120 (-0.000001) + 210 (0.00000001)$

$= 1 - 0.1 + 0.0045 - 0.00012 + 0.0000021$

≈ 0.9043821

$2^{10} = 1024$

$(1.98)^{10} \approx 1024 \times 0.9043821 \approx 926.087$, to three decimal places: 926.087

Answer: $(1.98)^{10} \approx 926.087$.

6. (b) The polynomial $x^5 + 4x^2 + ax + b$ leaves the remainder of $2x + 3$ when it is divided by $x^2 - 1$. Use the remainder theorem to find the values of a and b.

Let $p(x) = x^5 + 4x^2 + ax + b$, divisor $x^2 - 1 = (x - 1)(x + 1)$.

Remainder $= 2x + 3$ when divided by $x^2 - 1$, so:

$p(1) = 1 + 4 + a + b = 2(1) + 3 = 5$, $a + b = 0$

$p(-1) = (-1)^5 + 4(-1)^2 + a(-1) + b = -1 + 4 - a + b = 2(-1) + 3 = 1$, $-a + b + 3 = 1$, $-a + b = -2$

Solve: $a + b = 0$, $-a + b = -2$

$2b = -2$, $b = -1$

$a + (-1) = 0$, $a = 1$

Answer: $a = 1$, $b = -1$.

6. (c) The roots of the quadratic equation $x^2 + 2mx + n = 0$ differ by 2. Show that $m^2 = 1 + n$.

Roots are r and $r + 2$.

Sum of roots: $r + (r + 2) = 2r + 2 = -2m$ (from $-b/a$, where $b = 2m$, $a = 1$)

$2r + 2 = -2m$

$r + 1 = -m$

$r = -m - 1$

Product of roots: $r (r + 2) = n$

$(-m - 1) (-m - 1 + 2) = n$

$(-m - 1) (-m + 1) = n$

$m^2 - 1 = n$

$m^2 = 1 + n$

Answer: $m^2 = 1 + n$, as shown.

6. (d) If A = [[4 4 -1], [0 0 2], [m -1 1]] is singular, find the value of m.

det(A) = 0 for singular matrix:

det(A) = 4 (0 x 1 - 2 x (-1)) - 4 (0 x 1 - 2 x m) + (-1) (0 x (-1) - 0 x m)

= 4 (0 + 2) - 4 (0 - 2m) - 1 (0 - 0)

= 8 + 8m

8 + 8m = 0, m = -1

Answer: m = -1.

6. (e) Use Cramer's rule to solve the following system of equations:

$5x + 6y + 4z = 5$

$7x - 4y - 3z = 8$

$2x + 3y + 2z = 2$

det(A) = 5 (-4 x 2 - (-3) x 3) - 6 (7 x 2 - (-3) x 2) + 4 (7 x 3 - (-4) x 2)

= 5 (-8 + 9) - 6 (14 + 6) + 4 (21 + 8) = 5 - 120 + 116 = 1

det(A_x) = 5 (-4 x 2 - (-3) x 3) - 6 (8 x 2 - (-3) x 2) + 4 (8 x 3 - (-4) x 2) = 5 - 132 + 128 = 1

det(A_y) = 5 (2 x 2 - (-3) x 2) - 5 (7 x 2 - (-3) x 2) + 4 (7 x 2 - 2 x 2) = 50 - 100 + 48 = -2

det(A_z) = 5 (-4 x 2 - 8 x 3) - 6 (7 x 2 - 8 x 2) + 5 (7 x 3 - (-4) x 2) = -160 + 6 + 145 = -9

$x = \det(A_x) / \det(A) = 1/1 = 1$

$y = \det(A_y) / \det(A) = -2/1 = -2$

$z = \det(A_z) / \det(A) = -9/1 = -9$

Answer: $x = 1$, $y = -2$, $z = -9$.

7. (a) Form a differential equation whose solution is $x = \tan(Ay)$.

$x = \tan(Ay)$

$dx/dy = A \sec^2(Ay)$

$\sec^2(Ay) = 1 + \tan^2(Ay) = 1 + x^2$

$dx/dy = A (1 + x^2)$

$(1 + x^2)\, dx/dy = A$

$(1 + x^2)\, d^2x/dy^2 + 2x\, dx/dy = 0$

Answer: $(1 + x^2)\, d^2x/dy^2 + 2x\, dx/dy = 0$.

7. (b) Solve the differential equation $d^2\theta/dt^2 - 4\, d\theta/dt + 4\theta = 3/7$.

Homogeneous: $d^2\theta/dt^2 - 4\, d\theta/dt + 4\theta = 0$

$m^2 - 4m + 4 = 0$, $(m - 2)^2 = 0$, $m = 2$ (repeated root)

$\theta_h = (C + Dt)\, e^{2t}$

Particular: $\theta_p = K$ (constant, since RHS is constant)

$4K = 3/7$, $K = 3/28$

General: $\theta = (C + Dt)\, e^{2t} + 3/28$

Answer: $\theta = (C + Dt)\, e^{2t} + 3/28$.

7. (c) A biologist is researching the population of a species and solves them to compare with observed data. Her first model is $dn/dt = kr (1 - n/a)$ where n is the population at time t years, k is a constant, and a is the maximum population sustainable by the environment. Given that $k = 0.2$, $a = 100000$ and the initial population is 30000:

(i) Find the general solution of the differential equation.

$dn/dt = 0.2\, n (1 - n/100000)$

$dn / (n (1 - n/100000)) = 0.2\, dt$

Partial fractions: $1 / (n (1 - n/100000)) = A/n + B/(1 - n/100000)$

$A = 1/100000$, $B = 1/100000$

$(1/100000) (1/n + 1/(1 - n/100000)) dn = 0.2 dt$

$(1/100000) (\ln|n| - \ln|1 - n/100000|) = 0.2 t + C$

$\ln(n / (100000 - n)) = 20 t + C'$

$n / (100000 - n) = e^{(20 t + C')} = D e^{(20 t)}$

$n = D e^{(20 t)} (100000 - n)$

$n + D e^{(20 t)} n = 100000 D e^{(20 t)}$

$n (1 + D e^{(20 t)}) = 100000 D e^{(20 t)}$

$n = 100000 D e^{(20 t)} / (1 + D e^{(20 t)})$

Answer: $n = 100000 D e^{(20 t)} / (1 + D e^{(20 t)})$.

(ii) Estimate the population after 5 years to 2 significant figures.

At $t = 0$, $n = 30000$:

$30000 = 100000 D / (1 + D)$, $D = 3/7$

At $t = 5$: $n = 100000 (3/7) e^{(20 \times 5)} / (1 + (3/7) e^{(100)})$

e^{100} is very large, so $1 + (3/7) e^{100} \approx (3/7) e^{100}$

$n \approx 100000 (3/7) / (3/7) = 100000$

To 2 significant figures: 100000

Answer: Population ≈ 100000.

8. (a) Express $x^2 + y^2 = 2x + 2y$ in polar form.

$x = r \cos \theta$, $y = r \sin \theta$

$(r \cos \theta)^2 + (r \sin \theta)^2 = 2 (r \cos \theta + r \sin \theta)$

$r^2 (\cos^2 \theta + \sin^2 \theta) = 2r (\cos \theta + \sin \theta)$

$r^2 = 2r (\cos \theta + \sin \theta)$

$r = 2 (\cos \theta + \sin \theta)$

Answer: $r = 2 (\cos \theta + \sin \theta)$.

8. (b) Find the equation of the chord of the ellipse $x^2/a^2 + y^2/b^2 = 1$ joining the points whose eccentric angles are θ and φ.

Points: $(a \cos \theta, b \sin \theta)$ and $(a \cos \varphi, b \sin \varphi)$

Slope: $(b \sin \varphi - b \sin \theta) / (a \cos \varphi - a \cos \theta) = (b/a) (\sin \varphi - \sin \theta) / (\cos \varphi - \cos \theta)$

Using identities: $(\sin \varphi - \sin \theta) / (\cos \varphi - \cos \theta) = -\cot ((\varphi + \theta)/2)$

Slope = $-(b/a) \cot ((\varphi + \theta)/2)$

Equation using point $(a \cos \theta, b \sin \theta)$:

$(y - b \sin \theta) = -(b/a) \cot ((\varphi + \theta)/2) (x - a \cos \theta)$

Simplify (as needed):

$x / (a \cos ((\varphi + \theta)/2)) + y / (b \sin ((\varphi + \theta)/2)) = 1$ (standard form after simplification).

Answer: $x / (a \cos ((\varphi + \theta)/2)) + y / (b \sin ((\varphi + \theta)/2)) = 1$.

8. (c) Show that $P(a \sec \theta, b \tan \theta)$ lies on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, hence find the equation of the tangent line at point P on the given hyperbola.

$(a \sec \theta)^2 / a^2 - (b \tan \theta)^2 / b^2 = \sec^2 \theta - \tan^2 \theta = 1$, so P lies on the hyperbola.

$dy/dx = (b^2 x) / (a^2 y)$, at P: $dy/dx = (b^2 a \sec \theta) / (a^2 b \tan \theta) = (\sec \theta) / (a \tan \theta)$

Tangent: $y - b \tan \theta = (\sec \theta / (a \tan \theta)) (x - a \sec \theta)$

$x \sec \theta / a - y \tan \theta / b = 1$

Answer: Tangent: $x \sec \theta / a - y \tan \theta / b = 1$.

8. (d) Show whether the equation of a normal to the parabola $y^2 = 4ax$ at point (x_1, y_1) is $(x - x_1) y_1 + 2a (y - y_1) = 0$.

$y^2 = 4ax$, $dy/dx = 2a/y$, at (x_1, y_1): $dy/dx = 2a/y_1$

Normal slope: $-y_1/(2a)$

Normal: $y - y_1 = (-y_1/(2a)) (x - x_1)$

$(y - y_1) (2a) = -y_1 (x - x_1)$

$(x - x_1) y_1 + 2a (y - y_1) = 0$

Answer: The equation is correct, as shown.

8. (e) (i) Change the polar equation $r^2 (b^2 \cos^2 \theta + a^2 \sin^2 \theta) = a^2 b^2$ into the Cartesian equation.

$r^2 = x^2 + y^2$, $\cos\theta = x/r$, $\sin\theta = y/r$

$r^2 (b^2 (x/r)^2 + a^2 (y/r)^2) = a^2 b^2$

$(x^2 + y^2) (b^2 x^2 + a^2 y^2) / (x^2 + y^2) = a^2 b^2$

$b^2 x^2 + a^2 y^2 = a^2 b^2$

$b^2 x^2 / (a^2 b^2) + a^2 y^2 / (a^2 b^2) = 1$

$x^2 / a^2 + y^2 / b^2 = 1$

Answer: $x^2 / a^2 + y^2 / b^2 = 1$.

(ii) Draw the graph of $r = 2 (1 + \cos\theta)$.

$r = 2 (1 + \cos\theta)$ is a cardioid:

$\theta = 0$: $r = 4$

$\theta = \pi/2$: $r = 2$

$\theta = \pi$: $r = 0$

$\theta = 3\pi/2$: $r = 2$

Graph: Cardioid with cusp at (0, 0), extending to (4, 0).

Answer: Cardioid with cusp at origin, max at (4, 0).