THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 ADVANCED MATHEMATICS 2

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2022**

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) The time taken by John to deliver milk to the High Street is normally distributed with mean 12 minutes and standard deviation 2 minutes. If he delivers milk every day, estimate the number of days during the year when he takes longer than 17 minutes (1 year = 365 days).

Mean $\mu = 12$, SD $\sigma = 2$.

$P(X > 17) = P(Z > (17 - 12) / 2) = P(Z > 2.5)$

$P(Z > 2.5) \approx 1 - 0.9938 = 0.0062$

Number of days = $365 \times 0.0062 \approx 2.263$

Answer: Approximately 2 days.

1. (b) Suppose that a group of people in a village attending hospital has been categorized according to the incidence of two diseases:

Sex	Malaria	Typhoid
Male	16	12
Female	12	10

Find the probability that the person chosen is a female given the person is suffering from malaria.

Total malaria = 16 + 12 = 28

Female with malaria = 12

P(Female | Malaria) = 12 / 28 = 3 / 7

Answer: The probability is 3/7.

1. (c) In how many ways can a hand of 4 cards be chosen from an ordinary pack of 52 playing cards?

Number of ways = C(52, 4)

= (52 x 51 x 50 x 49) / (4 x 3 x 2 x 1)

= 6497400 / 24 = 270725

Answer: 270725 ways.

2. (a) Write the converse and inverse of the statement "If you score an A grade in a logic test, then I will buy you a new car" in words and symbolic form.

Statement: $p \rightarrow q$ (p: score an A, q: buy a new car)

Converse: $q \rightarrow p$

"If I buy you a new car, then you scored an A grade in a logic test."

Inverse: ¬p → ¬q

"If you do not score an A grade in a logic test, then I will not buy you a new car."

Answer: Converse: "If I buy you a new car, then you scored an A grade in a logic test" (q → p).

Inverse: "If you do not score an A grade in a logic test, then I will not buy you a new car" (¬p → ¬q).

2. (b) Using a truth table, examine whether [¬(p → (p ∧ q))] ⊻ (p → q) is equivalent to (q → p) ∧ (p → q).

Truth table:

p	q	p ∧ q	p → (p ∧ q)	¬(p → (p ∧ q))	p → q	[¬(p → (p ∧ q))] ⊻ (p → q)	q → p	(q → p) ∧ (p → q)
0	0	0	1	0	1	1	1	1
0	1	0	1	0	1	1	0	0
1	0	0	0	1	0	1	1	0
1	1	1	1	0	1	1	1	1

Columns [¬(p → (p ∧ q))] ⊻ (p → q) and (q → p) ∧ (p → q) differ, not equivalent.

Answer: Not equivalent.

2. (c) Use laws of algebra of propositions to simplify [p ∧ (p ∨ q)] ∨ [q ∧ (¬(p ∧ q))].

[p ∧ (p ∨ q)] ∨ [q ∧ (¬(p ∧ q))]

= p ∧ (p ∨ q) ∨ (q ∧ (¬p ∨ ¬q)) (De Morgan's)

= p ∨ (q ∧ ¬p ∧ ¬q) (Distributive, Absorption)

= p ∨ (q ∧ ¬q ∧ ¬p) (Commutative)

= p ∨ (0 ∧ ¬p) (Complement)

= p ∨ 0 = p

Answer: p.

3. (a) Find the work done by force F = i + 2j + k moving an object at a distance of 7 m in the direction of the vector r = 3i + 2j + 4k.

Unit vector in direction of r: $|r| = \sqrt{(3^2 + 2^2 + 4^2)} = \sqrt{(9 + 4 + 16)} = \sqrt{29}$

Unit vector = $(3/\sqrt{29}, 2/\sqrt{29}, 4/\sqrt{29})$

Force component: $F \cdot (r/|r|) = (1 \times 3/\sqrt{29}) + (2 \times 2/\sqrt{29}) + (1 \times 4/\sqrt{29}) = (3 + 4 + 4) / \sqrt{29} = 11 / \sqrt{29}$

Work = $(11 / \sqrt{29}) \times 7 = 77 / \sqrt{29}$

Answer: Work done = $77 / \sqrt{29}$ units.

3. (b) If P and Q are the points P(3, -4, 6) and Q(1, -3, 8) respectively, find a unit vector parallel to the displacement vector PQ.

PQ = Q - P = (1 - 3, -3 - (-4), 8 - 6) = (-2, 1, 2)

$|PQ| = \sqrt{(-2)^2 + 1^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$

Unit vector = (-2/3, 1/3, 2/3)

Answer: Unit vector = (-2/3, 1/3, 2/3).

3. (c) The position vectors of points A and B are a and b respectively. If point C divides AB internally in the ratio of 2:1, D divides AB externally in the ratio of 1:4 and E divides CD internally in the ratio of 2:1, find the position vectors of C, D, and E in terms of a and b.

C divides AB in 2:1: C = (1a + 2b) / (2 + 1) = (a + 2b) / 3

D divides AB externally in 1:4: D = (1b - 4a) / (1 - 4) = (-4a + b) / (-3) = (4a - b) / 3

E divides CD in 2:1: E = (1C + 2D) / (1 + 2) = ((a + 2b)/3 + 2 (4a - b)/3) / 3 = (a + 2b + 8a - 2b) / 9 = (9a / 9) = a

Answer: C = (a + 2b) / 3, D = (4a - b) / 3, E = a.

4. (a) Express $\sqrt{(1 + i)}$ in polar form.

$1 + i$: $r = \sqrt{(1^2 + 1^2)} = \sqrt{2}$, $\theta = \tan^{-1}(1/1) = \pi/4$

$1 + i = \sqrt{2} (\cos(\pi/4) + i \sin(\pi/4))$

$\sqrt{(1 + i)} = (\sqrt{2})^{(1/2)} (\cos(\pi/4 + 2k\pi)/2 + i \sin(\pi/4 + 2k\pi)/2)$

$= 2^{(1/4)} (\cos(\pi/8) + i \sin(\pi/8))$ (k = 0)

$= 2^{(1/4)} (\cos(5\pi/8) + i \sin(5\pi/8))$ (k = 1)

Answer: $2^{(1/4)} (\cos(\pi/8) + i \sin(\pi/8))$, $2^{(1/4)} (\cos(5\pi/8) + i \sin(5\pi/8))$.

4. (b) Using the results in part (a), show that $\tan(\pi/8) = \sqrt{2} - 1$.

$\sqrt{(1 + i)} = 2^{(1/4)} (\cos(\pi/8) + i \sin(\pi/8))$

Also, $\sqrt{(1 + i)} = (1 + i)^{(1/2)}$, let $(1 + i)^{(1/2)} = a + bi$

$(1 + i) = (a + bi)^2 = a^2 - b^2 + 2abi$

Real: $a^2 - b^2 = 1$

Imaginary: $2ab = 1$

Solve: $ab = 1/2$, $b = 1/(2a)$, substitute: $a^2 - (1/(2a))^2 = 1$

$a^2 - 1/(4a^2) = 1$

$4a^4 - 1 = 4a^2$

$4a^4 - 4a^2 - 1 = 0$

$a^2 = (4 \pm \sqrt{(16 + 16)}) / 8 = (4 \pm 4\sqrt{2}) / 8 = (1 \pm \sqrt{2}) / 2$

$a^2 = (1 + \sqrt{2}) / 2$, $a = \sqrt{((1 + \sqrt{2})/2)}$

$b = 1/(2a) = \sqrt{((\sqrt{2} - 1)/2)}$

$\tan(\pi/8) = b / a = \sqrt{((\sqrt{2} - 1)/(1 + \sqrt{2}))} = \sqrt{2} - 1$ (after rationalizing)

Answer: $\tan(\pi/8) = \sqrt{2} - 1$.

4. (c) If $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$, prove that $\mathrm{Arg}(z_1 / z_2) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$.

$z_1 / z_2 = (r_1 / r_2) (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$

$\mathrm{Arg}(z_1 / z_2) = \theta_1 - \theta_2 = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$

Answer: Proven.

4. (d) The complex numbers $z_1 = c / (1 + i)$ and $z_2 = d / (1 + 2i)$ where $c, d \in \mathbb{R}$ are such that $z_1 + z_2 = 1$, find the values of c and d.

$z_1 = c (1 - i) / (1 + i)(1 - i) = c (1 - i) / 2$

$z_2 = d (1 - 2i) / (1 + 2i)(1 - 2i) = d (1 - 2i) / 5$

$z_1 + z_2 = (c/2 - ci/2) + (d/5 - 2di/5) = 1$

Real: $c/2 + d/5 = 1$

Imaginary: $-c/2 - 2d/5 = 0$

From imaginary: $-c/2 = 2d/5$, $c = -4d/5$

Substitute: $(-4d/5)/2 + d/5 = 1$

$-2d/5 + d/5 = 1$

$-d/5 = 1$

$d = -5$

$c = -4(-5)/5 = 4$

Verify: $z_1 = 4(1 - i)/2 = 2 - 2i$, $z_2 = -5(1 - 2i)/5 = -1 + 2i$, $z_1 + z_2 = 1$

Answer: $c = 4$, $d = -5$.

5. (a) For all values of α, show that $\sin 3\alpha - \cos 3\alpha = 2$.

This appears incorrect as $\sin 3\alpha - \cos 3\alpha \neq 2$ for all α (e.g., $\alpha = 0$: $\sin 0 - \cos 0 = 0 - 1 = -1$). Likely a typo; let's assume the intended problem is to evaluate or prove a trigonometric identity. Let's try a similar identity or rephrase. Assume $\sin 3\alpha - \cos 3\alpha = 2 \sin \alpha \cos \alpha$ (as a possible correction).

$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$

$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$

$\sin 3\alpha - \cos 3\alpha = (3 \sin \alpha - 4 \sin^3 \alpha) - (4 \cos^3 \alpha - 3 \cos \alpha)$

This does not simplify to 2, so the problem statement is likely incorrect. Without a correct identity, we cannot proceed. Let's move to the next problem.

Answer: Problem statement appears incorrect; cannot be shown as given.

5. (b) Prove that $(\sin x + \sin 2x + \sin 3x) / (\cos x + \cos 2x + \cos 3x) = \tan 2x$.

Numerator: $\sin x + \sin 3x + \sin 2x$

$\sin x + \sin 3x = 2 \sin((x + 3x)/2) \cos((x - 3x)/2) = 2 \sin 2x \cos(-x) = 2 \sin 2x \cos x$

Add $\sin 2x$: $2 \sin 2x \cos x + \sin 2x = \sin 2x (2 \cos x + 1)$

Denominator: $\cos x + \cos 3x + \cos 2x$

$\cos x + \cos 3x = 2 \cos((x + 3x)/2) \cos((x - 3x)/2) = 2 \cos 2x \cos x$

Add $\cos 2x$: $2 \cos 2x \cos x + \cos 2x = \cos 2x (2 \cos x + 1)$

Expression: $[\sin 2x (2 \cos x + 1)] / [\cos 2x (2 \cos x + 1)] = \sin 2x / \cos 2x = \tan 2x$

Answer: Proven.

5. (c) Solve for β in the trigonometric equation $\tan^{-1}((\beta - 2)/(\beta - 2)) + \tan^{-1}((\beta + 1)/(\beta + 2)) = \pi/4$.

Since $(\beta - 2)/(\beta - 2) = 1$ for $\beta \neq 2$, first term is $\tan^{-1}(1) = \pi/4$.

Equation: $\pi/4 + \tan^{-1}((\beta + 1)/(\beta + 2)) = \pi/4$

$\tan^{-1}((\beta + 1)/(\beta + 2)) = 0$

$(\beta + 1)/(\beta + 2) = 0$

$\beta + 1 = 0$

$\beta = -1$

Check: $\tan^{-1}((-1 - 2)/(-1 - 2)) + \tan^{-1}((-1 + 1)/(-1 + 2)) = \pi/4 + \tan^{-1}(0) = \pi/4$, correct.

Answer: $\beta = -1$.

5. (d) Rewrite $4 \cos \theta - 3 \sin \theta$ in the form $R \cos(\theta - \alpha)$, hence solve the equation $4 \cos \theta - 3 \sin \theta = 5/2$ in the interval $\pi/2 \leq \theta \leq 2\pi$.

$R = \sqrt{(4^2 + (-3)^2)} = \sqrt{(16 + 9)} = \sqrt{25} = 5$

$\cos \alpha = 4/5$, $\sin \alpha = -3/5$

$4 \cos \theta - 3 \sin \theta = 5 (\cos \theta \cos \alpha + \sin \theta \sin \alpha) = 5 \cos(\theta - \alpha)$

$5 \cos(\theta - \alpha) = 5/2$

$\cos(\theta - \alpha) = 1/2$

$\theta - \alpha = \pm\pi/3 + 2k\pi$

$\alpha = \arccos(4/5)$, $\theta = \pm\pi/3 + \alpha + 2k\pi$

For $\pi/2 \leq \theta \leq 2\pi$, $k = 0$:

$\theta = \pi/3 + \alpha$ (≈ 1.97, within range)

$\theta = -\pi/3 + \alpha + 2\pi$ (≈ 5.71, within range)

Answer: $\theta = \pi/3 + \alpha$, $2\pi - \pi/3 + \alpha$, where $\cos \alpha = 4/5$.

6. (a) If the coefficients of x and x^2 in the expansion of $(1 + px + qx^2) / (1 - x)^2$ are zero, find the numerical values of p and q.

Expand $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$ (binomial expansion for $n = -2$)

$(1 + px + qx^2) / (1 - x)^2 = (1 + px + qx^2) (1 + 2x + 3x^2 + 4x^3 + \ldots)$

$= 1 + (p + 2)x + (q + 2p + 3)x^2 + (4p + 3q + 4)x^3 + \ldots$

Coefficient of x: $p + 2 = 0$, $p = -2$

Coefficient of x^2: $q + 2p + 3 = 0$, $q + 2(-2) + 3 = 0$, $q - 4 + 3 = 0$, $q = 1$

Answer: $p = -2$, $q = 1$.

6. (b) Use the principle of mathematical induction to prove that for every positive integer, 3^(2n-2) + 2^(6n) is divisible by 5.

Base case (n = 1):

3^(2(1)-2) + 2^(6(1)) = 3^0 + 2^6 = 1 + 64 = 65, 65 / 5 = 13, divisible by 5.

Assume true for n = k: 3^(2k-2) + 2^(6k) = 5m

For n = k + 1:

3^(2(k+1)-2) + 2^(6(k+1)) = 3^(2k) + 2^(6k+6)

= 9 x 3^(2k-2) + 64 x 2^(6k)

= 9 (5m - 2^(6k)) + 64 x 2^(6k) (using inductive hypothesis)

= 45m - 9 x 2^(6k) + 64 x 2^(6k)

= 45m + 55 x 2^(6k)

45m is divisible by 5, 55 x 2^(6k) is divisible by 5 (since 55 / 5 = 11), so the sum is divisible by 5.

Answer: Proven, 3^(2n-2) + 2^(6n) is divisible by 5 for all positive integers n.

6. (c) Given that $P(x) = 2x^3 + 7x^2 - 5$, use the synthetic method to find the quotient and remainder when P(x) is divided by x + 3.

Synthetic division: $x + 3 \rightarrow x = -3$

-3 | 2 7 -5

| -6 -3 24

| 2 1 -8 24

Quotient: $2x^2 + x - 8$

Remainder: 24

Answer: Quotient = $2x^2 + x - 8$, Remainder = 24.

6. (d) Find the determinant and inverse of the matrix A = [[2 1 0], [1 5 2], [1 -1 1]], hence solve the simultaneous equations:

$2x + y = 4$

$x + 5y + 2z = 7$

$x - y + z = 1$

det(A) = 2 (5 x 1 - 2 x (-1)) - 1 (1 x 1 - 2 x 1) + 0 = 2 (5 + 2) - 1 (1 - 2) = 14 + 1 = 15

Cofactors:

C_11 = 5 x 1 - 2 x (-1) = 7, C_12 = -(1 x 1 - 2 x 1) = 1, C_13 = (1 x (-1) - 5 x 1) = -6

C_21 = -(1 x 1 - 0 x (-1)) = -1, C_22 = (2 x 1 - 0 x 1) = 2, C_23 = -((2 x (-1) - 1 x 0)) = 2

C_31 = (1 x 2 - 5 x 0) = 2, C_32 = -((2 x 2 - 0 x 1)) = -4, C_33 = (2 x 5 - 1 x 1) = 9

Adjoint: [[7 -1 2], [1 2 -4], [-6 2 9]]

A^{-1} = (1/15) [[7 1 -6], [-1 2 2], [2 -4 9]]

Solve: X = A^{-1} B, B = [4, 7, 1]

x = (1/15) (7 x 4 + 1 x 7 + (-6) x 1) = (1/15) (28 + 7 - 6) = 39/15 = 13/5

y = (1/15) (-1 x 4 + 2 x 7 + 2 x 1) = (1/15) (-4 + 14 + 2) = 12/15 = 4/5

z = (1/15) (2 x 4 + (-4) x 7 + 9 x 1) = (1/15) (8 - 28 + 9) = -11/15

Answer: det(A) = 15, A^{-1} = (1/15) [[7 1 -6], [-1 2 2], [2 -4 9]], x = 13/5, y = 4/5, z = -11/15.

7. (a) (i) Determine the most general function M(x, y) such that the differential equation $M(x, y)\,dx + (2x^2 y^2 + x^2 y)\,dy = 0$ is exact.

For exactness: $\partial M/\partial y = \partial(2x^2 y^2 + x^2 y)/\partial x = 4x y^2 + 2x y$

$M = \int (4x y^2 + 2x y)\,dy = 4x y^3/3 + x y^2 + f(x)$

Answer: $M(x, y) = 4x y^3/3 + x y^2 + f(x)$.

7. (a) (ii) By separating the variables, solve the differential equation $(x + y^2)\,dx - (x^2 y^2 + x + y + 1)\,dy = 0$.

We are solving the differential equation:

$(xy + x)\,dx - (x^2y^2 + x^2 + y^2 + 1)\,dy = 0$

Write it as M(x, y) dx + N(x, y) dy = 0

$M(x, y) = x(y + 1)$

$N(x, y) = - (x^2y^2 + x^2 + y^2 + 1)$

Rewriting the equation:

$dy/dx = (x(y + 1)) / (x^2y^2 + x^2 + y^2 + 1)$

Factor the denominator

$x^2y^2 + x^2 + y^2 + 1$

$= x^2(y^2 + 1) + (y^2 + 1)$

$= (y^2 + 1)(x^2 + 1)$

Substitute the factorized denominator

$dy/dx = (x(y + 1)) / ((y^2 + 1)(x^2 + 1))$

Step 4: Separate variables

Multiply both sides by $(y^2 + 1)$ dy and multiply both sides by dx denominator to isolate variables

$(y^2 + 1)\, dy = (x(y + 1)) / (x^2 + 1)\, dx$

Now, move all terms involving y to one side and x to the other

$(y^2 + 1) / (y + 1)\, dy = (x) / (x^2 + 1)\, dx$

Simplify left side

Divide $(y^2 + 1)$ by $(y + 1)$ using polynomial division or factor if possible.

Let's divide:

$(y^2 + 1) \div (y + 1)$

$=y^2 + 1 = (y + 1)(y - 1) + 2$

So:

$(y^2 + 1) / (y + 1) = y - 1 + 2 / (y + 1)$

Now the separated equation is:

$(y - 1 + 2 / (y + 1))\, dy = (x) / (x^2 + 1)\, dx$

Integrate both sides

$\int (y - 1)\, dy + \int (2 / (y + 1))\, dy$

$= (1/2) y^2 - y + 2 \ln|y + 1|$

Integrating right side:

$\int x / (x^2 + 1)\, dx$

Let $u = x^2 + 1$, then $du = 2x\, dx$

So, $(1/2)\, du = x\, dx$

$\int x / (x^2 + 1)\, dx = (1/2) \int du / u = (1/2) \ln|x^2 + 1|$

$(1/2) y^2 - y + 2 \ln|y + 1| = (1/2) \ln|x^2 + 1| + C$

7. (b) Find the general solution of the differential equation $\cos x\, d^2y/dx^2 - \sin x\, dy/dx = 0$.

$\cos x\, d^2y/dx^2 - \sin x\, dy/dx = 0$

$d^2y/dx^2 - (\sin x / \cos x)\, dy/dx = 0$

Let $u = dy/dx$:

$du/dx - (\tan x) u = 0$

$du/u = \tan x\, dx$

$\ln|u| = -\ln|\cos x| + C$

$u = K / \cos x$

$dy/dx = K / \cos x$

$y = K \int (1/\cos x)\, dx + D = K \ln|\tan(x/2 + \pi/4)| + D$

Answer: $y = K \ln|\tan(x/2 + \pi/4)| + D$.

7. (c) A liquid of 72°C placed in a room at 25°C has a temperature of 65°C after 5 minutes. Find its temperature after further 10 minutes.

$dT/dt = -k (T - 25)$

$T - 25 = C e^{(-kt)}$

At $t = 0$, $T = 72$: $72 - 25 = C$, $C = 47$

At $t = 5$, $T = 65$: $65 - 25 = 47 e^{(-5k)}$, $40/47 = e^{(-5k)}$

$\ln(40/47) = -5k$, $k = -(1/5) \ln(40/47)$

At $t = 15$ (further 10 minutes): $T - 25 = 47 (40/47)^{(15/5)} = 47 (40/47)^3$

$(40/47)^3 \approx 0.617$

$T - 25 = 47 \times 0.617 \approx 29$

$T \approx 54$

Answer: Temperature ≈ 54°C.

7. (d) Formulate a differential equation of a circle which passes through the origin and whose centre lies on the y-axis.

Centre (0, k), passes through (0, 0):

$x^2 + (y - k)^2 = k^2$

$x^2 + y^2 - 2ky = 0$

Differentiate: $2x + 2y\, dy/dx - 2k\, dy/dx = 0$

$x + y\, dy/dx - k\, dy/dx = 0$

$k = (x + y\, dy/dx) / (dy/dx)$

Substitute into original: $x^2 + y^2 - 2 (x + y\, dy/dx) / (dy/dx)\, y = 0$ (simplify as needed).

Answer: $x^2 + y^2 - 2 (x + y\, dy/dx) / (dy/dx)\, y = 0$.

8. (a) Find the coordinates of the foci, the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $16x^2 - 9y^2 = 576$.

$16x^2 - 9y^2 = 576$

$x^2/36 - y^2/64 = 1$

$a = 6$, $b = 8$

Foci: $(\pm\sqrt{(a^2 + b^2)}, 0) = (\pm\sqrt{(36 + 64)}, 0) = (\pm 10, 0)$

Vertices: $(\pm a, 0) = (\pm 6, 0)$

Eccentricity: $e = \sqrt{(a^2 + b^2)} / a = 10/6 = 5/3$

Latus rectum: $2b^2/a = 2 \times 64 / 6 = 128/6 = 64/3$

Answer: Foci: $(\pm 10, 0)$, Vertices: $(\pm 6, 0)$, Eccentricity: 5/3, Latus rectum: 64/3.

8. (b) (i) Determine the equation of the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $P(a \cos \theta, b \sin \theta)$.

$dy/dx = -(b^2 x) / (a^2 y)$, at P: $dy/dx = -(b^2 a \cos \theta) / (a^2 b \sin \theta) = -(b/a) \cot \theta$

Normal slope = $(a/b) \tan \theta$

Normal: $y - b \sin \theta = (a/b) \tan \theta (x - a \cos \theta)$

Simplify: $(a x \sin \theta) / (\cos \theta) - (b y \cos \theta) / (\sin \theta) = a^2 - b^2$

Answer: $(a x \sin \theta) / (\cos \theta) - (b y \cos \theta) / (\sin \theta) = a^2 - b^2$.

8. (b) (ii) If the normal in part (b) (i) meets the x-axis at A and the y-axis at B, find the area of the triangle AOB where O is the origin.

A: $((a^2 - b^2) \cos \theta / a, 0)$

B: $(0, (b^2 - a^2) \sin \theta / b)$

Area = $(1/2) \mid (a^2 - b^2) \cos \theta / a \times (b^2 - a^2) \sin \theta / b \mid = (a^2 - b^2)^2 |\sin \theta \cos \theta| / (2ab)$

Answer: Area = $(a^2 - b^2)^2 |\sin \theta \cos \theta| / (2ab)$.

8. (c) Show that the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is $x - ty + at^2 = 0$.

$y^2 = 4ax$, $dy/dx = 2a/y$, at $(at^2, 2at)$: $dy/dx = 2a / (2at) = 1/t$

Tangent: $y - 2at = (1/t) (x - at^2)$

$ty - 2at^2 = x - at^2$

$x - ty + at^2 = 0$

Answer: Proven.

8. (d) (i) Change the Cartesian equation $(x^2 + y^2)^2 - 2xy (x^2 - y^2) = 0$ into a polar equation.

$x = r \cos \theta$, $y = r \sin \theta$

$(x^2 + y^2)^2 - 2xy(x^2 - y^2) = (r^2)^2 - 2(r\cos\theta\, r\sin\theta)(r^2\cos^2\theta - r^2\sin^2\theta) = r^4 - 2r^4\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta)$

$= r^4(1 - 2\sin\theta\cos\theta(\cos 2\theta)) = r^4(1 - \sin 2\theta\cos 2\theta) = 0$

$r = 0$ or $1 - \sin 2\theta\cos 2\theta = 0$ (simplify further if needed).

Answer: $r = 0$ or $1 - \sin 2\theta\cos 2\theta = 0$.

8. (d) (ii) Sketch the graph of $r = 1 - 2\cos\theta$ from $\theta = 0$ to $\theta = 2\pi$.

$r = 1 - 2\cos\theta$ (limaçon with a loop):

$\theta = 0$: $r = 1 - 2 = -1$

$\theta = \pi/2$: $r = 1 - 0 = 1$

$\theta = \pi$: $r = 1 + 2 = 3$

$\theta = 3\pi/2$: $r = 1 - 0 = 1$

Graph: Limaçon with inner loop, symmetric about x-axis.

Answer: Limaçon with an inner loop.