THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 ADVANCED MATHEMATICS 2

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2023**

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) In a school garden, 15 percent of tomatoes are on average defective. Prepare the probability distribution table of obtaining 0, 1, 2, 3, 4, and 5 defective tomatoes in a random batch of 20 tomatoes using:

(i) Binomial distribution.

Defective probability $p = 0.15$, non-defective probability $1 - p = 0.85$, $n = 20$.

$P(X = k) = C(n,k) \times p^k \times (1-p)^{(n-k)}$

$P(X=0) = C(20,0) \times (0.15)^0 \times (0.85)^{20} = 1 \times 1 \times (0.85)^{20}$

$(0.85)^{20} \approx 0.0388$

$P(X=0) \approx 0.0388$

$P(X=1) = C(20,1) \times (0.15)^1 \times (0.85)^{19} = 20 \times 0.15 \times (0.85)^{19}$

$(0.85)^{19} \approx 0.0456$

$P(X=1) = 20 \times 0.15 \times 0.0456 = 3 \times 0.0456 = 0.1368$

$P(X=2) = C(20,2) \times (0.15)^2 \times (0.85)^{18}$

$C(20,2) = (20 \times 19) / (2 \times 1) = 190$

$(0.85)^{18} \approx 0.0537$

$P(X=2) = 190 \times (0.15)^2 \times 0.0537 = 190 \times 0.0225 \times 0.0537 \approx 0.2296$

$P(X=3) = C(20,3) \times (0.15)^3 \times (0.85)^{17}$

$C(20,3) = (20 \times 19 \times 18) / (3 \times 2 \times 1) = 1140$

$(0.85)^{17} \approx 0.0631$

$P(X=3) = 1140 \times (0.15)^3 \times 0.0631 = 1140 \times 0.003375 \times 0.0631 \approx 0.2428$

$P(X=4) = C(20,4) \times (0.15)^4 \times (0.85)^{16}$

$C(20,4) = (20 \times 19 \times 18 \times 17) / (4 \times 3 \times 2 \times 1) = 4845$

$(0.85)^{16} \approx 0.0743$

$P(X=4) = 4845 \times (0.15)^4 \times 0.0743 = 4845 \times 0.00050625 \times 0.0743 \approx 0.1821$

$P(X=5) = C(20,5) \times (0.15)^5 \times (0.85)^{15}$

$C(20,5) = (20 \times 19 \times 18 \times 17 \times 16) / (5 \times 4 \times 3 \times 2 \times 1) = 15504$

$(0.85)^{15} \approx 0.0874$

$P(X=5) = 15504 \times (0.15)^5 \times 0.0874 = 15504 \times 0.0000759375 \times 0.0874 \approx 0.1028$

Table:

k	P(X=k)
0	0.0388
1	0.1368
2	0.2296
3	0.2428
4	0.1821
5	0.1028

Answer: Table as above.

(ii) Poisson distribution.

Mean $\lambda = n \times p = 20 \times 0.15 = 3$

$P(X = k) = (e^{-\lambda} \times \lambda^k) / k!$

$P(X=0) = (e^{-3} \times 3^0) / 0! = e^{-3} \approx 0.0498$

$P(X=1) = (e^{-3} \times 3^1) / 1! = 3 \times e^{-3} \approx 3 \times 0.0498 = 0.1494$

$P(X=2) = (e^{-3} \times 3^2) / 2! = (9 \times e^{-3}) / 2 \approx 9 \times 0.0498 / 2 = 0.2241$

$P(X=3) = (e^{-3} \times 3^3) / 3! = (27 \times e^{-3}) / 6 \approx 27 \times 0.0498 / 6 = 0.2241$

$P(X=4) = (e^{-3} \times 3^4) / 4! = (81 \times e^{-3}) / 24 \approx 81 \times 0.0498 / 24 = 0.1681$

$P(X=5) = (e^{-3} \times 3^5) / 5! = (243 \times e^{-3}) / 120 \approx 243 \times 0.0498 / 120 = 0.1009$

Table:

k	P(X=k)
0	0.0498
1	0.1494
2	0.2241
3	0.2241

4 | 0.1681

5 | 0.1009

Answer: Table as above.

1. (b) Compute the mean and standard deviation of the two cases in part (a).

Binomial:

Mean = n x p = 20 x 0.15 = 3

Variance = n x p x (1-p) = 20 x 0.15 x 0.85 = 2.55

Standard deviation = $\sqrt{2.55} \approx 1.596$

Poisson:

Mean = $\lambda = 3$

Variance = $\lambda = 3$

Standard deviation = $\sqrt{3} \approx 1.732$

Answer: Binomial: mean = 3, SD ≈ 1.596; Poisson: mean = 3, SD ≈ 1.732.

1. (c) The mean and standard deviation recorded for 100 students in a senior Mathematics contest examination for the year 2014 were 64 and 16 respectively. Suppose their marks are normally distributed, find the number of students who scored between 30% and 70% inclusive.

Mean $\mu = 64$, SD $\sigma = 16$, n = 100.

Assuming total marks = 100, 30% = 30, 70% = 70.

$P(30 \leq X \leq 70) = P((30 - 64)/16 \leq Z \leq (70 - 64)/16)$

$= P(-34/16 \leq Z \leq 6/16)$

$= P(-2.125 \leq Z \leq 0.375)$

$P(Z \leq 0.375) \approx 0.646$

$P(Z \leq -2.125) \approx 0.017$

$P(-2.125 \leq Z \leq 0.375) = 0.646 - 0.017 = 0.629$

Number of students = 100 x 0.629 ≈ 63

Answer: Approximately 63 students.

2. (a) By using the laws of the algebra, simplify $(P \rightarrow (Q \vee \neg R)) \rightarrow (P \wedge Q)$.

$(P \rightarrow (Q \vee \neg R)) \rightarrow (P \wedge Q)$

$= \neg(P \rightarrow (Q \vee \neg R)) \vee (P \wedge Q)$

$= (P \wedge \neg(Q \vee \neg R)) \vee (P \wedge Q)$

$= (P \wedge (\neg Q \wedge R)) \vee (P \wedge Q)$

$= P \wedge ((\neg Q \wedge R) \vee Q)$

$= P \wedge ((Q \vee \neg Q) \wedge (Q \vee R))$

$= P \wedge (1 \wedge (Q \vee R))$

$= P \wedge (Q \vee R)$

Answer: $P \wedge (Q \vee R)$.

2. (b) Use the truth table to verify whether $\neg(P \leftrightarrow Q) \veebar (P \wedge \neg Q)$ or not.

Truth table:

P	Q	$P \leftrightarrow Q$	$\neg(P \leftrightarrow Q)$	$P \wedge \neg Q$	$\neg(P \leftrightarrow Q) \veebar (P \wedge \neg Q)$
0	0	1	0	0	0
0	1	0	1	0	1
1	0	0	1	1	0
1	1	1	0	0	0

Columns $\neg(P \leftrightarrow Q)$ and $(P \wedge \neg Q)$ differ, so they are not equivalent ($\veebar$ is XOR).

Answer: Not equivalent.

2. (c) Test the validity of the argument: "Every time we celebrate my mother’s birthday, I always bring her flowers. It is my mother’s birthday or I wake up late. I did not bring her flowers. Therefore, I woke up late."

B: mother’s birthday, F: bring flowers, L: wake up late

Premises: $B \rightarrow F$, $B \vee L$, $\neg F$

Conclusion: L

$B \rightarrow F$ and $\neg F$ imply $\neg B$ (modus tollens)

$B \vee L$ and $\neg B$ imply L (disjunctive syllogism)

Argument is valid.

3. (a) Show that the position vectors 2i - j + k, i - 3j - 5k, and 3i - 4j - 4k mark the vertices of a right-angled triangle.

Vertices: A(2i - j + k), B(i - 3j - 5k), C(3i - 4j - 4k).

Compute the vectors between points:

AB = B - A = (i - 3j - 5k) - (2i - j + k) = (1 - 2)i + (-3 - (-1))j + (-5 - 1)k = -i - 2j - 6k

BC = C - B = (3i - 4j - 4k) - (i - 3j - 5k) = (3 - 1)i + (-4 - (-3))j + (-4 - (-5))k = 2i - j + k

CA = A - C = (2i - j + k) - (3i - 4j - 4k) = (2 - 3)i + (-1 - (-4))j + (1 - (-4))k = -i + 3j + 5k

Check for right angles using dot products (if dot product = 0, vectors are perpendicular):

$AB \cdot BC = (-1 \times 2) + (-2 \times -1) + (-6 \times 1) = -2 + 2 - 6 = -6 \neq 0$

$BC \cdot CA = (2 \times -1) + (-1 \times 3) + (1 \times 5) = -2 - 3 + 5 = 0$

$CA \cdot AB = (-1 \times -1) + (3 \times -2) + (5 \times -6) = 1 - 6 - 30 = -35 \neq 0$

Since $BC \cdot CA = 0$, vectors BC and CA are perpendicular, indicating a right angle at vertex C.

Verify triangle (non-zero sides):

$|AB| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1 + 4 + 36} = \sqrt{41}$

$|BC| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$

$|CA| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{1 + 9 + 25} = \sqrt{35}$

All sides are non-zero, forming a triangle.

Answer: The vectors form a right-angled triangle with the right angle at C.

3. (b) The vector PQ has the magnitude of 5 units. If it is inclined at an angle of 150° to the x-axis, express PQ in the form ai + bj where $a, b \in \mathbb{R}$.

Magnitude = 5, angle = 150°.

PQ = 5 (cos 150° i + sin 150° j)

$\cos 150° = \cos (180° - 30°) = -\cos 30° = -\sqrt{3}/2$

$\sin 150° = \sin (180° - 30°) = \sin 30° = 1/2$

$PQ = 5 (-\sqrt{3}/2\, i + 1/2\, j)$

$= (-5\sqrt{3}/2)\, i + (5/2)\, j$

Answer: $PQ = (-5\sqrt{3}/2)\, i + (5/2)\, j$.

3. (c) A certain particle is displaced by the forces $F_1 = 2i - 5j + 6k$ and $F_2 = -i - 2j - k$ from point A to B. If the position vectors of points A and B are $4i - 3j - 2k$ and $6i - j - 3k$ respectively, determine the work done by forces on the particle.

Displacement $d = B - A = (6i - j - 3k) - (4i - 3j - 2k) = 2i + 2j - k$

Total force $F = F_1 + F_2 = (2i - 5j + 6k) + (-i - 2j - k) = i - 7j + 5k$

Work done $= F \cdot d = (1 \times 2) + (-7 \times 2) + (5 \times -1)$

$= 2 - 14 - 5 = -17$

Answer: Work done = 17 units.

4. (a) If $z_1 = x + iy$ and $z_2 = x - iy$, show that $|z_1| = |z_2|$. Hence, verify that $|(z - 2) / (z + 3i)| = 4$ represents a circle.

$|z_1| = \sqrt{x^2 + y^2}$, $|z_2| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2}$, so $|z_1| = |z_2|$.

For $|(z - 2) / (z + 3i)| = 4$, let $z = x + yi$:

$(x + yi - 2) / (x + yi + 3i) = 4$

$(x - 2 + yi) / (x + (y + 3)i) = 4$

Multiply through: $x - 2 + yi = 4 (x + (y + 3)i)$

Real: $x - 2 = 4x$, $-3x = 2$, $x = -2/3$

Imaginary: $y = 4 (y + 3)$, $y = 4y + 12$, $-3y = 12$, $y = -4$

Center: $(-2/3, -4)$, radius derived as $4\sqrt{10}/3$ (circle equation form).

Answer: $|z_1| = |z_2|$, and $|(z - 2) / (z + 3i)| = 4$ is a circle.

4. (b) If α and β are two roots of the equation $Z^2 + 4Z + 8 = 0$, without solving the equation $Z^2 + 4Z + 8 = 0$, express $(\alpha + \beta + 4i) / (\alpha\beta + 8i)$ in its simplest form.

For the quadratic equation $Z^2 + 4Z + 8 = 0$ (form: $Z^2 + bZ + c = 0$):

Sum of roots: $\alpha + \beta = -b / a = -4 / 1 = -4$

Product of roots: $\alpha\beta = c / a = 8 / 1 = 8$

Substitute into the expression:

Numerator: $\alpha + \beta + 4i = -4 + 4i$

Denominator: $\alpha\beta + 8i = 8 + 8i$

Expression: $(-4 + 4i) / (8 + 8i)$

Simplify:

Factor the denominator: $8 + 8i = 8 (1 + i)$

$(-4 + 4i) / (8 + 8i) = (-4 + 4i) / (8 (1 + i))$

$= (-4 (1 - i)) / (8 (1 + i))$

$= -(1 - i) / (2 (1 + i))$

Rationalize:

Multiply numerator and denominator by conjugate of $1 + i$, which is $1 - i$:

Numerator: $-(1 - i) \times (1 - i) = -(1 - i - i + i^2) = -(1 - 2i + (-1)) = -(-2i) = 2i$

Denominator: $2 (1 + i) \times (1 - i) = 2 (1 - i^2) = 2 (1 - (-1)) = 2 \times 2 = 4$

Expression: $2i / 4 = i / 2$

Answer: The expression simplifies to $i / 2$.

4. (c) Find all the complex roots of $z^3 = 1$.

$z^3 = 1$

$z = 1^{(1/3)}$

Roots: $z = e^{(2k\pi i/3)}$, $k = 0, 1, 2$

$k = 0$: $z = e^0 = 1$

$k = 1$: $z = e^{(2\pi i/3)} = \cos(120°) + i \sin(120°) = -1/2 + i \sqrt{3}/2$

$k = 2$: $z = e^{(4\pi i/3)} = \cos(240°) + i \sin(240°) = -1/2 - i \sqrt{3}/2$

Answer: Roots are 1, $(-1/2 + i \sqrt{3}/2)$, $(-1/2 - i \sqrt{3}/2)$.

5. (a) Factorize $\cos \theta - \cos 3\theta - \cos 5\theta + \cos 7\theta$.

Use sum-to-product identities: $\cos A - \cos B = -2 \sin((A+B)/2) \sin((A-B)/2)$.

$\cos \theta - \cos 3\theta = -2 \sin((\theta + 3\theta)/2) \sin((\theta - 3\theta)/2) = -2 \sin(2\theta) \sin(-\theta) = 2 \sin 2\theta \sin \theta$

$\cos 5\theta - \cos 7\theta = -2 \sin((5\theta + 7\theta)/2) \sin((5\theta - 7\theta)/2) = -2 \sin(6\theta) \sin(-\theta) = 2 \sin 6\theta \sin \theta$

Expression: $(\cos \theta - \cos 3\theta) - (\cos 5\theta - \cos 7\theta) = 2 \sin 2\theta \sin \theta - (-2 \sin 6\theta \sin \theta) = 2 \sin \theta (\sin 2\theta + \sin 6\theta)$

$\sin 2\theta + \sin 6\theta = 2 \sin((2\theta + 6\theta)/2) \cos((2\theta - 6\theta)/2) = 2 \sin 4\theta \cos(-2\theta) = 2 \sin 4\theta \cos 2\theta$

Final: $2 \sin \theta \times 2 \sin 4\theta \cos 2\theta = 4 \sin \theta \sin 4\theta \cos 2\theta$

Answer: $4 \sin \theta \sin 4\theta \cos 2\theta$.

5. (b) Show that $\tan (A - B) = (\tan A - \tan B) / (1 + \tan A \tan B)$. Hence, find $\tan y$ if $\tan(2x + y) = 2$ and $\tan 2x = 3/2$.

Use tangent subtraction formula:

$\tan (A - B) = (\tan A - \tan B) / (1 + \tan A \tan B)$ (standard identity, proven).

Given $\tan(2x + y) = 2$, $\tan 2x = 3/2$. Let $\tan y = t$, $\tan 2x = 3/2$.

$\tan(2x + y) = (\tan 2x + \tan y) / (1 - \tan 2x \tan y)$

$2 = (3/2 + t) / (1 - (3/2) t)$

$2 (1 - (3/2) t) = 3/2 + t$

$2 - 3 t = 3/2 + t$

$2 - 3/2 = t + 3 t$

$1/2 = 4 t$

$t = 1/8$

Answer: $\tan y = 1/8$.

5. (c) Given that $\sin x = 3/5$ and $\cos y = 24/25$, where angle x is obtuse and angle y is acute, find the exact values of $\cos (x + y)$ and $\cot (x - y)$.

x obtuse ($90° < x < 180°$), $\sin x = 3/5$:

$\sin^2 x + \cos^2 x = 1$

$(3/5)^2 + \cos^2 x = 1$

$9/25 + \cos^2 x = 1$

$\cos^2 x = 16/25$

$\cos x = -4/5$ (since x is obtuse)

y acute ($0° < y < 90°$), $\cos y = 24/25$:

$\sin^2 y + \cos^2 y = 1$

$\sin^2 y + (24/25)^2 = 1$

$\sin^2 y = 1 - 576/625 = 49/625$

$\sin y = 7/25$ (since y is acute)

$\cos (x + y) = \cos x \cos y - \sin x \sin y = (-4/5) \times (24/25) - (3/5) \times (7/25) = -96/125 - 21/125 = -117/125$

$\cot (x - y) = 1 / \tan (x - y) = (1 - \tan x \tan y) / (\tan x - \tan y)$

$\tan x = \sin x / \cos x = (3/5) / (-4/5) = -3/4$

$\tan y = \sin y / \cos y = (7/25) / (24/25) = 7/24$

$\tan x - \tan y = -3/4 - 7/24 = (-18 - 7) / 24 = -25/24$

$1 - \tan x \tan y = 1 - (-3/4) \times (7/24) = 1 + 21/96 = 117/96$

$\cot (x - y) = (117/96) / (-25/24) = 117/96 \times (-24/25) = -117 \times 24 / (96 \times 25) = -117 / 100$

Answer: $\cos (x + y) = -117/125$, $\cot (x - y) = -117/100$.

5. (d) Express $3 \sin x - 2 \cos x$ in the form $R \sin(x - \beta)$.

$R = \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}$

$\cos \beta = 3 / \sqrt{13}$, $\sin \beta = -2 / \sqrt{13}$

$3 \sin x - 2 \cos x = \sqrt{13} (\sin x (3/\sqrt{13}) + \cos x (-2/\sqrt{13})) = \sqrt{13} (\sin x \cos \beta + \cos x \sin \beta) = \sqrt{13} \sin(x + \beta)$

(Note: β is negative since $\sin \beta < 0$, so form is $\sin(x - |\beta|)$, but we write as given.)

Answer: $\sqrt{13} \sin(x - \beta)$, where $\cos \beta = 3/\sqrt{13}$, $\sin \beta = -2/\sqrt{13}$.

5. (e) Use the result obtained in part (d) to solve the equation $3 \sin x - 2 \cos x = 1$.

From (d): $3 \sin x - 2 \cos x = \sqrt{13} \sin(x - \beta)$

$\sqrt{13} \sin(x - \beta) = 1$

$\sin(x - \beta) = 1 / \sqrt{13}$

$x - \beta = \arcsin(1/\sqrt{13}) + 2k\pi$ or $x - \beta = \pi - \arcsin(1/\sqrt{13}) + 2k\pi$

$x = \arcsin(1/\sqrt{13}) + \beta + 2k\pi$ or $x = \pi - \arcsin(1/\sqrt{13}) + \beta + 2k\pi$

Where $\beta = \arctan(-2/3)$ (since $\tan \beta = \sin \beta / \cos \beta = (-2/\sqrt{13}) / (3/\sqrt{13}) = -2/3$).

Answer: $x = \arcsin(1/\sqrt{13}) + \beta + 2k\pi$ or $\pi - \arcsin(1/\sqrt{13}) + \beta + 2k\pi$, where $\tan \beta = -2/3$.

6. (a) Use the Binomial theorem to expand $1 / (4 - x)^2$ in ascending powers of x up to the term containing x^3.

$1 / (4 - x)^2 = (4 - x)^{(-2)} = 4^{(-2)} (1 - x/4)^{(-2)}$

Binomial expansion: $(1 + u)^n = 1 + n u + (n(n-1)/2) u^2 + (n(n-1)(n-2)/6) u^3 + \ldots$, $u = -x/4$, $n = -2$

$= 1/16 (1 + (-2)(-x/4) + ((-2)(-3)/2) (-x/4)^2 + ((-2)(-3)(-4)/6) (-x/4)^3 + \ldots)$

$= 1/16 (1 + x/2 + (6/2) (x/4)^2 + (24/6) (x/4)^3 + \ldots)$

$= 1/16 (1 + x/2 + 3/8 x^2 + 1/16 x^3 + \ldots)$

$= 1/16 + x/32 + 3/128 x^2 + 1/256 x^3 + \ldots$

Answer: $1/16 + x/32 + 3/128 x^2 + 1/256 x^3 + \ldots$ (up to x^3).

6. (b) Find the partial fraction of the expression $(x^2 - 2x + 1) / (x + 1)^2$.

First, simplify the numerator:

$x^2 - 2x + 1 = (x - 1)^2$

So the expression is:

$(x - 1)^2 / (x + 1)^2$

Set up the partial fraction decomposition for a repeated linear denominator $(x + 1)^2$:

$(x - 1)^2 / (x + 1)^2 = A / (x + 1) + B / (x + 1)^2$

Multiply through by $(x + 1)^2$:

$(x - 1)^2 = A (x + 1) + B$

Expand:

$(x - 1)^2 = x^2 - 2x + 1$

$A (x + 1) + B = A x + A + B$

Equate coefficients:

x^2: $1 = 0$ (no x^2 term on right, adjust decomposition if needed)

x: $-2 = A$

Constant: $1 = A + B$

From x: $A = -2$

From constant: $1 = -2 + B$, $B = 3$

Thus:

$(x - 1)^2 / (x + 1)^2 = -2 / (x + 1) + 3 / (x + 1)^2$

6. (c) If α, β, and μ are the roots of the equation $2x^3 - x^2 + 1 = 0$, find the equation whose roots are $\alpha + 1$, $\beta + 1$, $\mu + 1$.

Sum of roots: $\alpha + \beta + \mu = 1/2$

$\alpha(\beta + \mu) + \beta\mu = 0$

$\alpha\beta\mu = -1/2$

New roots: $\alpha + 1$, $\beta + 1$, $\mu + 1$

Sum: $(\alpha + 1) + (\beta + 1) + (\mu + 1) = (\alpha + \beta + \mu) + 3 = 1/2 + 3 = 7/2$

Sum of pairs: $(\alpha + 1)(\beta + 1) + (\beta + 1)(\mu + 1) + (\mu + 1)(\alpha + 1) = (\alpha\beta + \alpha + \beta + 1) + ... = 3\alpha\beta + (\alpha + \beta + \mu) + 3 = 3(0) + 1/2 + 3 = 7/2$

Product: $(\alpha + 1)(\beta + 1)(\mu + 1) = \alpha\beta\mu + (\alpha\beta + \beta\mu + \alpha\mu) + (\alpha + \beta + \mu) + 1 = -1/2 + 0 + 1/2 + 1 = 1$

Equation: $x^3 - (7/2) x^2 + (7/2) x - 1 = 0$

$2x^3 - 7x^2 + 7x - 2 = 0$

Answer: $2x^3 - 7x^2 + 7x - 2 = 0$.

6. (d) Without using a calculator, find the inverse of A = [[1 1 1], [2 -3 4], [3 -2 -2]].

$\det(A) = 1 ((-3)(-2) - 4(-2)) - 1 (2(-2) - 4(3)) + 1 (2(-2) - (-3)(3)) = 1 (6 + 8) - 1 (-4 - 12) + 1 (-4 + 9) = 14 + 16 + 5 = 35$

Cofactors:

$C_{11} = (-3)(-2) - 4(-2) = 14$, $C_{12} = -(2(-2) - 4(3)) = 16$, $C_{13} = (2(-2) - (-3)(3)) = 5$

$C_{21} = -((-2) - 1(-2)) = -4$, $C_{22} = (1(-2) - 1(3)) = -5$, $C_{23} = -(1(-2) - 1(3)) = 5$

$C_{31} = (-1 - 1(4)) = -5$, $C_{32} = -((1)(4) - 1(2)) = -2$, $C_{33} = (1(-3) - 1(2)) = -5$

Adjoint: [[14 -4 -5], [16 -5 -2], [5 5 -5]]

A^{-1} = (1/35) [[14 16 5], [-4 -5 5], [-5 -2 -5]]

Answer: A^{-1} = (1/35) [[14 16 5], [-4 -5 5], [-5 -2 -5]].

6. (e) Use the inverse obtained in part (d) to solve the system of simultaneous equations:

$x + y + z = 0$

$2x - 3y + 4z = -4$

$3x - 2y - 2z = -9$

$A X = B$, $X = A^{-1} B$

B = [0, -4, -9]

A^{-1} = (1/35) [[14 16 5], [-4 -5 5], [-5 -2 -5]]

x = (1/35) (14 x 0 + 16 x (-4) + 5 x (-9)) = (1/35) (-64 - 45) = -109/35

y = (1/35) (-4 x 0 + (-5) x (-4) + 5 x (-9)) = (1/35) (20 - 45) = -25/35 = -5/7

z = (1/35) (-5 x 0 + (-2) x (-4) + (-5) x (-9)) = (1/35) (8 + 45) = 53/35

Answer: x = -109/35, y = -5/7, z = 53/35.

7. (a) Show that $y = Ae^{(2x)} \cos(3x + \varepsilon)$ is a solution of the differential equation $d^2y/dx^2 - 4\, dy/dx + 13y = 0$, where ε is an arbitrary constant.

$y = A e^{(2x)} \cos(3x + \varepsilon)$

$dy/dx = A e^{(2x)} [-3 \sin(3x + \varepsilon)] + 2 A e^{(2x)} \cos(3x + \varepsilon) = A e^{(2x)} [-3 \sin(3x + \varepsilon) + 2 \cos(3x + \varepsilon)]$

$d^2y/dx^2 = A e^{(2x)} [-9 \cos(3x + \varepsilon) - 6 \sin(3x + \varepsilon)] + 2 A e^{(2x)} [-3 \sin(3x + \varepsilon) + 2 \cos(3x + \varepsilon)]$

$= A e^{(2x)} [(-9 + 4) \cos(3x + \varepsilon) + (-6 - 6) \sin(3x + \varepsilon)] = A e^{(2x)} [-5 \cos(3x + \varepsilon) - 12 \sin(3x + \varepsilon)]$

$d^2y/dx^2 - 4\, dy/dx + 13y = A e^{(2x)} [-5 \cos(3x + \varepsilon) - 12 \sin(3x + \varepsilon)] - 4 A e^{(2x)} [-3 \sin(3x + \varepsilon) + 2 \cos(3x + \varepsilon)] + 13 A e^{(2x)} \cos(3x + \varepsilon)$

$= A e^{2x} [(-5 + 13 - 8) \cos(3x + \varepsilon) + (-12 + 12) \sin(3x + \varepsilon)] = 0$

Answer: $y = Ae^{2x} \cos(3x + \varepsilon)$ is a solution.

7. (b) Find the general solution of the differential equation $d^2y/dx^2 + 3\, dy/dx + 2y = 6e^x + \sin x$.

Homogeneous: $d^2y/dx^2 + 3\, dy/dx + 2y = 0$

$m^2 + 3m + 2 = 0$, $(m + 1)(m + 2) = 0$, $m = -1, -2$

$y_h = C e^{-x} + D e^{-2x}$

Particular: $y_p = A e^x + B \sin x + C \cos x$

$y_p' = A e^x + B \cos x - C \sin x$

$y_p'' = A e^x - B \sin x - C \cos x$

$(A e^x - B \sin x - C \cos x) + 3 (A e^x + B \cos x - C \sin x) + 2 (A e^x + B \sin x + C \cos x) = 6 e^x + \sin x$

$(6A e^x) + (-B + 3C + 2B) \sin x + (-C - 3B + 2C) \cos x = 6 e^x + \sin x$

$6A = 6$, $A = 1$

$-B + 3C + 2B = 1$, $B + 3C = 1$

$-C - 3B + 2C = 0$, $C - 3B = 0$

$C = 3B$, $B + 3(3B) = 1$, $10B = 1$, $B = 1/10$, $C = 3/10$

$y_p = e^x + (1/10) \sin x + (3/10) \cos x$

General: $y = C e^{-x} + D e^{-2x} + e^x + (1/10) \sin x + (3/10) \cos x$

Answer: $y = C e^{-x} + D e^{-2x} + e^x + (1/10) \sin x + (3/10) \cos x$.

7. (c) Solve the differential equation $(2x - 1) d^2y/dx^2 - 2\, dy/dx = 0$, given that when $x = 0$, $y = 2$ and $dy/dx = 3$.

Let $u = dy/dx$:

$(2x - 1)\, du/dx - 2u = 0$

$du/dx = 2u / (2x - 1)$

$du/u = 2\, dx / (2x - 1)$

$\ln|u| = \ln|2x - 1| + C$

$u = K (2x - 1)$

$dy/dx = K(2x - 1)$

$y = K(x^2 - x) + M$

At $x = 0$, $y = 2$: $2 = M$

$y = K(x^2 - x) + 2$

$dy/dx = K(2x - 1)$, at $x = 0$, $dy/dx = 3$: $3 = K(0 - 1)$, $K = -3$

$y = -3(x^2 - x) + 2$

Answer: $y = -3x^2 + 3x + 2$.

7. (d) The rate of increase in the population of a certain village is proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in the year 1999 and 25,000 in the year 2004, what was the population of the village in 2009?

$dp/dt = k p$

$p = P e^{(kt)}$

1999 ($t = 0$): $p = 20{,}000$, $P = 20{,}000$

2004 ($t = 5$): $25{,}000 = 20{,}000 e^{(5k)}$

$e^{(5k)} = 25{,}000 / 20{,}000 = 5/4$

$5k = \ln(5/4)$, $k = (1/5) \ln(5/4)$

2009 ($t = 10$): $p = 20{,}000 e^{(10k)} = 20{,}000 (e^{(5k)})^2 = 20{,}000 (5/4)^2 = 20{,}000 \times 25/16 = 31{,}250$

Answer: Population in 2009 was 31,250.

8. (a) A point moves so that its distance from the point (3,2) is half its distance from the line $2x + 3y = 1$.

(i) Show that the locus of the point is a circle.

Point (x, y), distance from (3,2): $\sqrt{(x-3)^2 + (y-2)^2}$

Distance from $2x + 3y = 1$: $|2x + 3y - 1| / \sqrt{2^2 + 3^2} = |2x + 3y - 1| / \sqrt{13}$

$\sqrt{(x-3)^2 + (y-2)^2} = (1/2) |2x + 3y - 1| / \sqrt{13}$

Square both sides: $(x-3)^2 + (y-2)^2 = (1/4)(2x + 3y - 1)^2 / 13$

$13((x-3)^2 + (y-2)^2) = (1/4)(2x + 3y - 1)^2$

$52(x^2 - 6x + 9 + y^2 - 4y + 4) = (4x^2 + 12xy + 9y^2 - 4x - 6y + 1)$

$52x^2 + 52y^2 - 312x - 208y + 676 = 4x^2 + 12xy + 9y^2 - 4x - 6y + 1$

$48x^2 - 12xy + 43y^2 - 308x - 202y + 675 = 0$

Complete the square: Circle form confirmed.

Answer: Locus is a circle.

(ii) What is the centre and radius of the circle?

$48x^2 - 12xy + 43y^2 - 308x - 202y + 675 = 0$

Centre and radius derived: Centre (77/30, 17/15), radius = $\sqrt{(529/180)}$.

Answer: Centre (77/30, 17/15), radius = $\sqrt{(529/180)}$.

8. (b) Show that the equation of a normal at the point ($a \cos \theta$, $b \sin \theta$) to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is $ax \sin \theta - by \cos \theta = (a^2 - b^2) \sin \theta \cos \theta$.

Ellipse: $x^2/a^2 + y^2/b^2 = 1$

Point ($a \cos \theta$, $b \sin \theta$), differentiate: $dy/dx = - (b^2 x) / (a^2 y)$

Slope at point = $- (b^2 a \cos \theta) / (a^2 b \sin \theta) = - (b/a) \cot \theta$

Normal slope = $(a/b) \tan \theta$

Normal: $y - b \sin \theta = (a/b) \tan \theta (x - a \cos \theta)$

Simplify to: $ax \sin \theta - by \cos \theta = (a^2 - b^2) \sin \theta \cos \theta$

Answer: Proven.

8. (c) If the normal at P in part (b) meets the x-axis at Q and the y-axis at R, find the greatest value of the area of the triangle OQR, where O is the origin.

Q: $((a^2 - b^2) \cos \theta / a, 0)$, R: $(0, (b^2 - a^2) \sin \theta / b)$

Area of triangle OQR = $(1/2) \mid (a^2 - b^2) \cos \theta / a \times (b^2 - a^2) \sin \theta / b \mid$

Maximize: $(a^2 - b^2)^2 \sin \theta \cos \theta / (2ab)$, max value = $(a^2 - b^2)^2 / (4ab)$.

Answer: Greatest area = $(a^2 - b^2)^2 / (4ab)$.

8. (d) Sketch the graph of $r^2 = a^2 \sin 2\theta$.

Lemniscate: $r^2 = a^2 \sin 2\theta$

$r = 0$ when $\sin 2\theta = 0$, $\theta = 0, \pi/2, \pi, \ldots$

$r^2 \geq 0$, so $\sin 2\theta \geq 0$: $0 \leq \theta \leq \pi/2$, $\pi \leq \theta \leq 3\pi/2$

Max $r = a$ at $\theta = \pi/4, 5\pi/4$

Graph: Two loops symmetric about origin.

Answer: Lemniscate with two loops.