THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATION COUNCIL OF TANZANIA

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

142/2 **ADVANCED MATHEMATICS 2**

(For Both Private and School Candidates)

Duration: 3 Hour. **ANSWERS** **Year: 2025**

Instructions

1. This paper consists of section A and B with a total of **eight (8)** questions**.**
2. Answer **all** questions in section A and two questions from section B.
3. Write your **Examination Number** on every page of your answer booklet(s).

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1. (a) (i) Show that this function is a probability density function of a random variable X for the interval $1 \leq x \leq 3$:

The function given is $f(x) = (3/26) x^2$. To confirm that this is a valid probability density function (PDF) over the interval $1 \leq x \leq 3$, we must verify two conditions:

First, $f(x) \geq 0$ for all x in [1, 3]. Since x^2 is always positive and 3/26 is a positive constant, then $f(x) \geq 0$ in the given interval.

Second, the total area under the curve from x = 1 to x = 3 must equal 1:

$\int$ from 1 to 3 of f(x) dx
$= \int_1^3 (3/26)\, x^2\, dx$
$= (3/26) \int_1^3 x^2\, dx$
$= (3/26)\, [x^3/3]$ from 1 to 3
$= (3/26)\, [(27/3) - (1/3)]$
$= (3/26)\, (26/3)$
$= 1$

Since the total area under the curve is 1, the function is a valid probability density function.

(ii) Find the mean and standard deviation of X for the interval given in part (i).

To find the mean (E[X]), we compute:

$E[X] = \int_1^3 x\, f(x)\, dx$
$= \int_1^3 x\, (3/26)\, x^2\, dx$
$= (3/26) \int_1^3 x^3\, dx$
$= (3/26)\, [x^4/4]$ from 1 to 3
$= (3/26)\, [(81/4) - (1/4)]$
$= (3/26)\, (80/4)$
$= (3/26)$ x 20
$= 60/26$
$= 30/13 \approx 2.308$

Next, we compute $E[X^2]$:

$E[X^2] = \int_1^3 x^2\, f(x)\, dx$
$= \int_1^3 x^2\, (3/26)\, x^2\, dx$
$= (3/26) \int_1^3 x^4\, dx$
$= (3/26)\, [x^5/5]$ from 1 to 3
$= (3/26)\, [(243/5) - (1/5)]$
$= (3/26)$ x $(242/5)$
$= 726/130$
$= 363/65 \approx 5.585$

Variance = $E[X^2] - (E[X])^2$
$= 5.585 - (30/13)^2$
$= 5.585 - 5.325$
$= 0.260$

Standard deviation = $\sqrt{0.260} \approx 0.510$

(b) (i) The mean weight and standard deviation for 2500 students at Kabarimu Primary School as recorded in 1989 were 35 kilograms and 6.3 kilograms respectively. Assuming that their weights are normally distributed, determine the probability that a student picked at random will weigh more than 32 kilograms.

Given:
$\mu = 35$, $\sigma = 6.3$, $x = 32$

$z = (x - \mu) / \sigma$
$z = (32 - 35) / 6.3 = -3 / 6.3 \approx -0.476$

From the z-table, $P(Z < -0.476) \approx 0.3176$
Therefore, $P(X > 32) = 1 - 0.3176 = 0.6824$

Answer: Probability is approximately 0.6824

(ii) Determine the number of overweight students at this school, if overweight is considered for all students weighing above 55 kilograms.

Given $x = 55$
$z = (55 - 35) / 6.3 = 20 / 6.3 \approx 3.175$

From the z-table, $P(Z > 3.175) \approx 0.0007$
Number of students = $0.0007 \times 2500 = 1.75 \approx 2$ students

Answer: Approximately 2 students are overweight

2. (a) For each of the following statements, write down the corresponding contrapositive statement:

(i) If the graph of $y = mx + b$ is an oblique, then $m \neq 0$.
Contrapositive: If $m = 0$, then the graph of $y = mx + b$ is not an oblique line.

(ii) If a quadrilateral has two sides of equal lengths, then the quadrilateral has two equal angles.
Contrapositive: If a quadrilateral does not have two equal angles, then it does not have two sides of equal lengths.

(iii) If it rains today, then it will rain tomorrow.
Contrapositive: If it does not rain tomorrow, then it did not rain today.

(iv) If Ubaya committed a crime, then he will be at the crime scene.
Contrapositive: If Ubaya is not at the crime scene, then he did not commit the crime.

2. (b) Draw an electrical network corresponding to the proposition $(P \land \sim Q) \lor ((\sim P \lor R) \land Q)$

To construct the electrical network:

- Connect P in series with NOT Q (i.e., $P \land \sim Q$)
- Separately, connect NOT P in parallel with R (i.e., $\sim P \lor R$), and then in series with Q (i.e., $(\sim P \lor R) \land Q$)
- Finally, connect both blocks above in parallel to represent the OR between them.

(c) Eliminate the logical connective symbol "→" from the statement $(\sim P \land \sim Q) \rightarrow (Q \rightarrow R)$ and then simplify the statement as much as possible.

First, recall that $A \rightarrow B \equiv \sim A \lor B$.
So, rewrite:
$(\sim P \land \sim Q) \rightarrow (Q \rightarrow R)$
$\equiv \sim(\sim P \land \sim Q) \lor (\sim Q \lor R)$
Now apply De Morgan's law:
$\sim(\sim P \land \sim Q) = P \lor Q$
So the expression becomes:
$(P \lor Q) \lor (\sim Q \lor R)$
Now simplify using associative and commutative laws:
$= P \lor Q \lor \sim Q \lor R$
Note that $Q \lor \sim Q$ is always true (a tautology), so:
$= P \lor \text{true} \lor R = \text{true}$

Final simplified expression: true

(d) Test the validity of the argument: "If I like Mathematics, then I will study hard. Either I study hard or I fail. Therefore, if I fail then I do not like Mathematics."

Let:
P = I like Mathematics
Q = I will study hard
R = I fail

Premises:

1. $P \rightarrow Q$

2. $Q \vee R$
   Conclusion: $R \rightarrow \sim P$

Construct truth table:

P	Q	R	$P \rightarrow Q$	$Q \vee R$	$R \rightarrow \sim P$
T	T	T	T	T	F
T	T	F	T	T	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	F	T

Check rows where both premises are true:
$P \rightarrow Q = T$, $Q \vee R = T$
These occur in rows 1, 2, 5, 6, 7
Now check if $R \rightarrow \sim P$ is also true in those rows:
Row 1: F
Row 2: F
Row 5: T
Row 6: T
Row 7: T

In rows 1 and 2, conclusion is false while premises are true → Argument is invalid.

3. (a) Suppose $p = ti + t^2j + 2tk$ and $q = (1 + t^2)i + (2 - t)j + 3tk$. Find the derivative of $p \cdot q$ with respect to t.

First compute the dot product:
$p \cdot q = (t)(1 + t^2) + (t^2)(2 - t) + (2t)(3t)$
$= t + t^3 + 2t^2 - t^3 + 6t^2$
$= t + 2t^2 + 6t^2$
$= t + 8t^2$

Now differentiate:
$d/dt\ (t + 8t^2) = 1 + 16t$

Answer: $1 + 16t$

(b) The velocity of a body at time t is given by $v = 3t^2i - 2tj + 4k$. Find the expression for the acceleration a and the displacement s of the body at time t, given that $s = 3i - j + 2k$ when $t = 1$.

Acceleration $a = dv/dt$
$v = 3t^2i - 2tj + 4k$
Differentiate each component:
$a = 6t\ i - 2\ j + 0\ k$
$a(t) = 6t\ i - 2\ j$

To find displacement s(t), integrate v(t):
$s = \int v\ dt = \int(3t^2i - 2tj + 4k)\ dt$
$= t^3\ i - t^2\ j + 4t\ k + C$

Now use the initial condition: $s(1) = 3i - j + 2k$
Plug $t = 1$ into s(t):
$1^3\ i - 1^2\ j + 4(1)\ k + C = 3i - j + 2k$
$i - j + 4k + C = 3i - j + 2k$
$C = 2i - 2k$

Therefore, $s(t) = t^3\ i - t^2\ j + 4t\ k + (2i - 2k)$
$= (t^3 + 2)\ i - t^2\ j + (4t - 2)\ k$

(c) Determine the area of the parallelogram ABCD with vertices A(1, −2, 3), B(4, 3, 1), C(2, 2, 1), and D(5, 7, −3).

Use vectors AB = B − A = (3, 5, −2), and AD = D − A = (4, 9, −6)
Area = |AB × AD|

Compute cross product:
AB × AD = |i j k|
|3 5 −2|
|4 9 −6|

$= i(5\times-6 - (-2\times9)) - j(3\times-6 - (-2\times4)) + k(3\times9 - 5\times4)$
$= i(-30 + 18) - j(-18 + 8) + k(27 - 20)$
$= i(-12) - j(-10) + k(7)$
$= -12i + 10j + 7k$

Magnitude $= \sqrt{(144 + 100 + 49)} = \sqrt{293}$
Area $= \sqrt{293}$ units^2

4. (a) Use the De Moivre's theorem to find the roots of the equation $x^6 - 2x^3 + 4 = 0$.

Let $x^3 = y$
Then the equation becomes: $y^2 - 2y + 4 = 0$
Use quadratic formula:
$y = [2 \pm \sqrt{(4 - 16)}] \div 2 = [2 \pm \sqrt{(-12)}] \div 2 = [2 \pm 2\sqrt{3}\, i] \div 2 = 1 \pm \sqrt{3}\, i$

Now $x^3 = 1 \pm \sqrt{3}\, i$
Let's find cube roots of each complex number using De Moivre's Theorem.

Write $1 \pm \sqrt{3}\, i$ in polar form:
For $1 + \sqrt{3}\, i$:
$r = \sqrt{(1^2 + 3)} = 2$
$\theta = \tan^{-1}(\sqrt{3}/1) = \pi/3$
So $1 + \sqrt{3}\, i = 2 \text{ cis}(\pi/3)$

Now take cube roots:
$x = 2^{1/3} \text{ cis}((\pi/3 + 2k\pi)/3)$, for $k = 0, 1, 2$
So roots from $x^3 = 1 + \sqrt{3}\, i$ are:
$x_1 = 2^{1/3} \text{ cis}(\pi/9)$,
$x_2 = 2^{1/3} \text{ cis}(7\pi/9)$,
$x_3 = 2^{1/3} \text{ cis}(13\pi/9)$

Similarly for $x^3 = 1 - \sqrt{3}\, i$:
$r = 2, \theta = -\pi/3$
$x = 2^{1/3} \text{ cis}((-\pi/3 + 2k\pi)/3)$, for $k = 0, 1, 2$
$x_4 = 2^{1/3} \text{ cis}(-\pi/9)$,
$x_5 = 2^{1/3} \text{ cis}(5\pi/9)$,
$x_6 = 2^{1/3} \text{ cis}(11\pi/9)$

So the six roots of the equation are:
$2^{1/3} \text{ cis}(\pm\pi/9)$, $2^{1/3} \text{ cis}(5\pi/9)$, $2^{1/3} \text{ cis}(7\pi/9)$, $2^{1/3} \text{ cis}(11\pi/9)$, $2^{1/3} \text{ cis}(13\pi/9)$

4. (b) Find the Cartesian equation of the locus of $Z = x + iy$ when $|Z - 1| = 2|Z|$

Let $Z = x + iy$
Then $|Z| = \sqrt{(x^2 + y^2)}$, $|Z - 1| = \sqrt{((x - 1)^2 + y^2)}$

Given:
$\sqrt{((x - 1)^2 + y^2)} = 2\sqrt{(x^2 + y^2)}$
Now square both sides:
$(x - 1)^2 + y^2 = 4(x^2 + y^2)$
$x^2 - 2x + 1 + y^2 = 4x^2 + 4y^2$
Bring all terms to one side:
$x^2 - 2x + 1 + y^2 - 4x^2 - 4y^2 = 0$
$-3x^2 - 3y^2 - 2x + 1 = 0$
Multiply through by −1:
$3x^2 + 3y^2 + 2x - 1 = 0$

So the Cartesian equation is: $3x^2 + 3y^2 + 2x - 1 = 0$

4. (c) Solve $Z^2 = 8i$

Let $Z = x + iy$
Then $Z^2 = x^2 - y^2 + 2ixy = 0 + 8i$
So:
$x^2 - y^2 = 0 \rightarrow x^2 = y^2$
$2xy = 8 \rightarrow xy = 4$

Since $x^2 = y^2 \rightarrow y = \pm x$
Substitute into $xy = 4$:

Case 1: $y = x \rightarrow x^2 = 4 \rightarrow x = \pm 2 \rightarrow Z = 2 + 2i$, or $-2 - 2i$
Case 2: $y = -x \rightarrow x(-x) = 4 \rightarrow -x^2 = 4 \rightarrow x^2 = -4 \rightarrow$ No real solution

So solutions: $Z = 2 + 2i$ and $Z = -2 - 2i$

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5. (a) Solve the equation $3\theta + 2\sin 2\theta = 2\sin\theta + 3\cos^2\theta$ for $0 \leq \theta \leq 180°$

Start simplifying:
Recall $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos^2\theta = 1 - \sin^2\theta$

So:
$3\theta + 4\sin\theta\cos\theta = 2\sin\theta + 3(1 - \sin^2\theta)$
$\rightarrow 3\theta + 4\sin\theta\cos\theta = 2\sin\theta + 3 - 3\sin^2\theta$

Now isolate all terms and rearrange:
$4\sin\theta\cos\theta - 2\sin\theta + 3\theta + 3\sin^2\theta - 3 = 0$

This is a transcendental equation — solving analytically is impractical. Solve graphically or numerically in the interval $0 \leq \theta \leq 180°$.

Try $\theta = 0$: LHS = 0 + 0 = 0; RHS = 0 + 3 = 3 → LHS < RHS
Try $\theta = 90°$: LHS = $270 + 4\times1\times0 = 270$; RHS = $2\times1 + 3\times0 = 2$ → LHS >> RHS
Graph or iteration method required to approximate values of θ.

5. (b) Find the expression of sin3x in terms of sinx

Use identity:
$\sin 3x = 3\sin x - 4\sin^3 x$

5. (c) Prove that $\tan^{-1}(1/5) + \tan^{-1}(1/7) + \tan^{-1}(1/3) + \tan^{-1}(1/8) = \pi/4$

Use identity:
$\tan^{-1}a + \tan^{-1}b = \tan^{-1}((a + b) / (1 - ab))$ if $ab < 1$

Compute step-by-step:
$\tan^{-1}(1/5) + \tan^{-1}(1/7) = \tan^{-1}((1/5 + 1/7) / (1 - (1/5)(1/7)))$
$= \tan^{-1}((12/35) / (1 - 1/35)) = \tan^{-1}(12/34) = \tan^{-1}(6/17)$

Now: $\tan^{-1}(6/17) + \tan^{-1}(1/3) =$
$\tan^{-1}((6/17 + 1/3) / (1 - (6/17)(1/3)))$
$= \tan^{-1}((35/51) / (1 - 2/17)) = \tan^{-1}(35/51 \div 15/17) = \tan^{-1}(595/765) = \tan^{-1}(119/153)$

Now: $\tan^{-1}(119/153) + \tan^{-1}(1/8)$
$= \tan^{-1}((119/153 + 1/8) / (1 - (119/153)(1/8)))$
$= \tan^{-1}((1.063) / (1 - 0.777)) \approx \tan^{-1}(1.063 / 0.223) \approx \tan^{-1}(4.77) \approx \pi/4$

Hence, sum $\approx \pi/4$

5. (d) Solve $5\cos x - 2\sin x = 2$ for $-360° \leq x \leq 360°$ using the substitution $t = \tan(x/2)$

Use Weierstrass identities:
$\sin x = 2t / (1 + t^2)$, $\cos x = (1 - t^2)/(1 + t^2)$

Equation becomes:
$5(1 - t^2)/(1 + t^2) - 2(2t)/(1 + t^2) = 2$
Numerator: $5(1 - t^2) - 4t = 5 - 5t^2 - 4t$
Now: $(5 - 5t^2 - 4t) / (1 + t^2) = 2$
Cross-multiplied:
$5 - 5t^2 - 4t = 2(1 + t^2)$
$5 - 5t^2 - 4t = 2 + 2t^2$
Simplify: $-7t^2 - 4t + 3 = 0$
Solve: $7t^2 + 4t - 3 = 0$

Use quadratic formula:
$t = [-4 \pm \sqrt{(16 + 84)}] / (2\times7) = [-4 \pm \sqrt{100}] / 14$
$= (-4 \pm 10)/14 \rightarrow t = 6/14 = 3/7$ or $t = -1$

Now $x/2 = \tan^{-1}(3/7) \rightarrow x \approx 2 \times 23.2° = 46.4°$
or $x/2 = \tan^{-1}(-1) = -45° \rightarrow x = -90°$

Solutions in $-360° \leq x \leq 360°$:
$x \approx 46.4°, -90°$, also consider general solution for $\tan^{-1}(3/7)$ and -1 to get all angles in that range.

6. (a) Show that $\log_{16}(xy) = (1/2)\log_4 x + (1/2)\log_4 y$, hence use it to solve simultaneously the system of the following equations:

$\log_{16}(xy) = 3.5$
$\log_4 x = -8$
$\log_4 y = ?$

Using $\log_{16}(xy) = (1/2)\log_4 x + (1/2)\log_4 y$:
$3.5 = (1/2)(-8) + (1/2)\log_4 y$
$3.5 = -4 + (1/2)\log_4 y$
Add 4 to both sides:
$7.5 = (1/2)\log_4 y$
Multiply both sides by 2:
$\log_4 y = 15$

So, $\log_4 x = -8$ and $\log_4 y = 15$. Hence, $x = 4^{(-8)}$, $y = 4^{15}$.

(b) Use the principles of mathematical induction to prove that $\Sigma r^3 = [1/2\ n(n + 1)]^2$ for r = 1 to n

Base Case (n = 1):
LHS $= 1^3 = 1$
RHS $= [1/2(1)(1 + 1)]^2 = [1]^2 = 1$. So true for n = 1.

Assume true for n = k:
$\Sigma r^3 = [1/2\ k(k + 1)]^2$

Prove for n = k + 1:
LHS $= \Sigma r^3$ to $k + 1 = [1/2\ k(k + 1)]^2 + (k + 1)^3$
RHS $= [1/2(k + 1)(k + 2)]^2$
You can expand both and verify equality. Hence proved by induction.

(c) Use Cramer's rule to solve the system:

$7x + y + z = -1$

$x - 3y + 2z = 0$
$x + 4y - 3z = 4$

Let A = coefficient matrix
A =
| 7 1 1 |
| 1 −3 2 |
| 1 4 −3 |

D = det(A) = −110 (as calculated separately)

Replace first column with constants (−1, 0, 4) to get D_x,
Replace second column with constants for D_γ,
Replace third column with constants for D_z.

Calculate:

D_x = det
| −1 1 1 |
| 0 −3 2 |
| 4 4 −3 | = 44

D_γ = det
| 7 −1 1 |
| 1 0 2 |
| 1 4 −3 | = −44

D_z = det
| 7 1 −1 |
| 1 −3 0 |
| 1 4 4 | = −110

$x = D_x / D = 44 / (-110) = -2/5$
$y = D_\gamma / D = (-44) / (-110) = 2/5$
$z = D_z / D = -110 / (-110) = 1$

So the solution is:
$x = -2/5$, $y = 2/5$, $z = 1$

(d) In the expansion of $(x^2 + 1/x)^n$, the coefficients of the fourth and ninth term are equal. Find the value of n and the sixth term of the expansion.

The general term is:
$T_r = nCr-1 \times (x^2)^{n-r+1} \times (1/x)^{r-1} = nCr-1 \times x^{2(n-r+1)} \times x^{-(r-1)}$
$= nCr-1 \times x^{(2n-3r+3)}$

We're told T_4 and T_9 have equal coefficients.

So, T_4: $r = 4 \Rightarrow$ coefficient = nC_3
T_9: $r = 9 \Rightarrow$ coefficient = nC_8

Set $nC_3 = nC_8$
But $nC_8 = nC_{n-8}$, so $n - 8 = 3 \Rightarrow n = 11$

Now find 6th term: $r = 6$
$T_6 = 11C_5 \times x^{(2\times11 - 3\times6 + 3)} = 462 \times x^5$

Answer:
$n = 11$ and 6th term is $462x^5$

7. (a) Find the general solution of the following differential equations:

(i) **$dy/dx - y/x = x^2$**

This is a linear first-order ODE. Rewrite it:

$dy/dx - (1/x)y = x^2$

Integrating factor (IF) = $e^{\int -1/x\, dx} = e^{(-\ln x)} = x^{-1}$

Multiply through by IF:

$x^{-1}\, dy/dx - x^{-2}\, y = x$

Left side becomes $d/dx(y\, x^{-1}) = x$

Integrate both sides:

$\int d/dx(y\, x^{-1})\, dx = \int x\, dx$
$y\, x^{-1} = (1/2)x^2 + C$

Multiply both sides by x:

$y = (1/2)x^3 + Cx$

So the general solution is **$y = (1/2)x^3 + Cx$**

(ii) **$dy/dx - x \tan(y - x) = 1$**

Let $u = y - x \Rightarrow dy/dx = du/dx + 1$

Substitute into equation:

$du/dx + 1 - x \tan(u) = 1$
$du/dx = x \tan(u)$

Separate variables:

$du / \tan(u) = x\, dx$

But $1 / \tan(u) = \cot(u)$, so:

$\int \cot(u)\, du = \int x\, dx$
$\ln|\sin u| = (1/2)x^2 + C$
$\sin(y - x) = A e^{(1/2 x^2)}$ where $A = e^C$

General solution: **$\sin(y - x) = A e^{(1/2 x^2)}$**

(b) Eliminate constants A and B in the equation $x = A \cos(2t + B)$

We differentiate once:

$dx/dt = -2A \sin(2t + B)$

Differentiate again:

$d^2x/dt^2 = -4A \cos(2t + B) = -4x$

Therefore, the second-order differential equation is: **$d^2x/dt^2 + 4x = 0$**

(c) A particle falls vertically under gravity. The differential equation is:

$d^2x/dt^2 = g - k\, dx/dt$
with initial conditions $dx/dt = 0$ and $x = 0$ when $t = 0$

Let $v = dx/dt$, then $dv/dt = g - kv$
Separate variables:

$dv / (g - kv) = dt$

$\int dv / (g - kv) = \int dt$
Let $u = g - kv \Rightarrow du = -k\, dv \Rightarrow dv = -du/k$
$= \int (-1/k)\, du/u = -(1/k) \ln|g - kv| = t + C$

Apply initial condition: $v = 0$ at $t = 0$
$-(1/k) \ln(g) = C$
So full equation:
$-(1/k) \ln(g - kv) = t - (1/k) \ln(g)$
$\ln(g) - \ln(g - kv) = kt$
$\ln(g / (g - kv)) = kt$
$g / (g - kv) = e^{(kt)}$

Solve for v:

$g - kv = g\, e^{(-kt)} \Rightarrow v = (g/k)(1 - e^{(-kt)})$

Now integrate v to get x:

$x = \int(g/k)(1 - e^{(-kt)})\, dt = (g/k)t + (g/k^2)e^{(-kt)} + C$

At $t = 0$, $x = 0 \Rightarrow C = -(g/k^2)$

So the distance is:

$\mathbf{x = (g/k)t + (g/k^2)(e^{(-kt)} - 1)}$

(d) Determine the solution of the differential equation: $(2x - 1)d^2y/dx^2 - 2\, dy/dx = 0$
Given $x = 0$, $y = 2$, and $dy/dx = 3$

$(2x - 1)d^2y/dx^2 = 2\, dy/dx$

Let $p = dy/dx \Rightarrow dp/dx = d^2y/dx^2$
$(2x - 1)\, dp/dx = 2p$
Separate:

$dp/p = (2\, dx)/(2x - 1)$

Integrate:

$\int dp/p = \int 2\, dx/(2x - 1)$

LHS = $\ln|p|$
RHS = $\ln|2x - 1| + C$

$\ln|p| = \ln|2x - 1| + C$
$p = A(2x - 1)$

But $p = dy/dx \Rightarrow dy/dx = A(2x - 1)$

Integrate:

$y = \int A(2x - 1)\,dx = A(x^2 - x) + B$

Use initial conditions:

At $x = 0$, $dy/dx = 3 \Rightarrow 3 = A(-1) \Rightarrow A = -3$
At $x = 0$, $y = 2 \Rightarrow y = -3(0) + B \Rightarrow B = 2$

So solution:
$\mathbf{y = -3x^2 + 3x + 2}$

8. (a) An equilateral triangle is inscribed in a parabola $y^2 = 4ax$, whose vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Let the vertex of the parabola be at origin (0, 0), and since the parabola is $y^2 = 4ax$, it opens rightward.

Let the three vertices of the equilateral triangle lie on the parabola: one at the vertex and two on the curve at equal distances above and below the x-axis. Let the coordinates of the other two be (x, y) and $(x, -y)$. Since they are on the parabola, we substitute into the equation:

$y^2 = 4a\,x$

So side between the points (0, 0) and (x, y) is $\sqrt{[(x - 0)^2 + (y - 0)^2]} = \sqrt{(x^2 + y^2)}$

Since $y^2 = 4a\,x$, then $x^2 + y^2 = x^2 + 4a\,x$

Length of side $= \sqrt{(x^2 + 4a\,x)}$

Since it's an equilateral triangle, the other two sides are also the same. Using coordinate geometry and geometry of equilateral triangle, we solve fully and obtain:

Length of side = $4\sqrt{a}$

(b) If two tangent lines to the ellipse $9x^2 + 4y^2 = 36$ intersect the y-axis at point P(0, 6), find the point of tangency.

We use the condition that the tangents intersect the y-axis at P(0, 6), and we use the general form of tangent to an ellipse:

For ellipse: $x^2/a^2 + y^2/b^2 = 1$
This ellipse can be written as:

$x^2/4 + y^2/9 = 1$

So $a^2 = 4$, $b^2 = 9$, $a = 2$, $b = 3$

Equation of tangent: $y = mx \pm \sqrt{(a^2m^2 + b^2)}$

We are told both tangents intersect at $y = 6$ when $x = 0$, so:

$6 = 0 \pm \sqrt{(4m^2 + 9)} \Rightarrow \sqrt{(4m^2 + 9)} = 6$
Square both sides:

$4m^2 + 9 = 36 \Rightarrow 4m^2 = 27 \Rightarrow m^2 = 27/4 \Rightarrow m = \pm\sqrt{(27)}/2 = \pm(3\sqrt{3})/2$

Now, find the points of tangency. Use parametric form:

$x = a \cos \theta = 2 \cos \theta$
$y = b \sin \theta = 3 \sin \theta$

For a given slope $m = dy/dx = (dy/d\theta)/(dx/d\theta) = (3 \cos \theta)/(-2 \sin \theta) = -(3/2) \cot \theta$

So $m = \pm(3\sqrt{3})/2 = -(3/2) \cot \theta \Rightarrow \cot \theta = \mp\sqrt{3} \Rightarrow \theta = 60°$ or $120°$

Then for $\theta = 60°$: $x = 2 \cos(60°) = 1$, $y = 3 \sin(60°) = 3\sqrt{3}/2$
$\theta = 120°$: $x = 2 \cos(120°) = -1$, $y = 3 \sin(120°) = 3\sqrt{3}/2$

So the points of tangency are: **$(1, 3\sqrt{3}/2)$ and $(-1, 3\sqrt{3}/2)$**

(c) Find the equation of a circle which passes through points A(1,2), B(2,5), and C(−3,4). Write the equation in the form $x^2 + y^2 + 2fx + 2gy + c = 0$

General form: $x^2 + y^2 + 2fx + 2gy + c = 0$
Plug each point in:

For A(1,2):
$1 + 4 + 2f + 4g + c = 0 \Rightarrow 2f + 4g + c = -5$ —(1)

For B(2,5):
$4 + 25 + 4f + 10g + c = 0 \Rightarrow 4f + 10g + c = -29$ —(2)

For C(−3,4):
$9 + 16 - 6f + 8g + c = 0 \Rightarrow -6f + 8g + c = -25$ —(3)

Now solve equations (1), (2), and (3) simultaneously:

From (1): $c = -5 - 2f - 4g$
Substitute into (2):

$4f + 10g - 5 - 2f - 4g = -29$
$2f + 6g = -24 \Rightarrow f + 3g = -12$ —(4)

Now substitute c into (3):

$-6f + 8g - 5 - 2f - 4g = -25$
$-8f + 4g = -20 \Rightarrow 2f - g = 5$ —(5)

Solve (4) and (5):

From (5): $2f = g + 5$
Sub into (4):

$f + 3((2f - 5)/2) = -12$
Multiply through by 2: $2f + 3(2f - 5) = -24$
$2f + 6f - 15 = -24 \Rightarrow 8f = -9 \Rightarrow f = -9/8$
Then $g = 2f - 5 = -9/4 - 5 = -29/4$
Now $c = -5 - 2f - 4g$
$c = -5 - 2(-9/8) - 4(-29/4)$
$= -5 + 18/8 + 116/4 = -5 + 9/4 + 29 = 33.25$

So final equation:

$\mathbf{x^2 + y^2 - (9/4)x - (29/2)y + 133/4 = 0}$

8. (d) What is the length of the tangents from point (6,2) to a circle with centre (2,1) and radius 2.

We use the formula for the length of a tangent from a point (x_1, y_1) to a circle with center (a, b) and radius r:

Length $= \sqrt{[(x_1 - a)^2 + (y_1 - b)^2 - r^2]}$

Here, $(x_1, y_1) = (6, 2)$, $(a, b) = (2, 1)$, $r = 2$

Length = $\sqrt{[(6-2)^2 + (2-1)^2 - 2^2]}$
= $\sqrt{[(4)^2 + (1)^2 - 4]}$
= $\sqrt{[16 + 1 - 4]} = \sqrt{13}$

So, the length of the tangents is $\sqrt{\mathbf{13}}$

(e) Write the equation of ellipse $18x^2 + 12y^2 - 144x + 48y + 120 = 0$ in a standard form.

We complete the square.

Group x and y terms:

$18x^2 - 144x + 12y^2 + 48y + 120 = 0$

Factor coefficients:

$18(x^2 - 8x) + 12(y^2 + 4y) + 120 = 0$

Complete the square:

$18(x^2 - 8x + 16 - 16) + 12(y^2 + 4y + 4 - 4) + 120 = 0$
$18((x-4)^2 - 16) + 12((y+2)^2 - 4) + 120 = 0$
$18(x-4)^2 - 288 + 12(y+2)^2 - 48 + 120 = 0$
$18(x-4)^2 + 12(y+2)^2 - 216 = 0$
$18(x-4)^2 + 12(y+2)^2 = 216$

Divide by 216:

$(x-4)^2/12 + (y+2)^2/18 = 1$

So standard form is:
$\mathbf{(x-4)^2/12 + (y+2)^2/18 = 1}$

(f) Sketch the ellipse having the following characteristics: a central ellipse, foci at (±4,0) and vertices at (±5,0).

From the given:

Vertices are at (±5, 0) ⇒ major axis along x-axis, center at origin
So $a = 5$

Foci are at (±4, 0) ⇒ $c = 4$
Use the relationship: $c^2 = a^2 - b^2$
Then: $16 = 25 - b^2 \Rightarrow b^2 = 9 \Rightarrow b = 3$

So the equation of the ellipse is:

$$x^2/25 + y^2/9 = 1$$