THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

141 BASIC APPLIED MATHEMATICS

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2006**

Instructions

1. This paper consists of **Ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. Boy scouts stationed at camp C, in figure 1 below, wish to cross a river to point B

THEY HAVE FOUND THAT ANGLE ABC = 31°06'. POINT A IS A TOILET STATIONED 34.76 m FROM CAMP C.

ANGLE BCA = 112°53'. How wide is the river between B and C?

In triangle ABC, we are given:

Angle ABC = 31°06' = 31 + (6/60) = 31.1°

Angle BCA = 112°53' = 112 + (53/60) = 112 + 0.8833 = 112.8833°

Side AC = 34.76 m

First, find angle BAC:

Angle BAC = 180° - Angle ABC - Angle BCA = 180° - 31.1° - 112.8833° = 36.0167°

Use the Law of Sines to find side BC (the width of the river):

(BC / sin(BAC)) = (AC / sin(ABC))

BC / sin(36.0167°) = 34.76 / sin(31.1°)

BC = 34.76 × (sin(36.0167°) / sin(31.1°))

Calculate the sines:

sin(36.0167°) ≈ 0.5878

sin(31.1°) ≈ 0.5170

BC = 34.76 × (0.5878 / 0.5170) ≈ 34.76 × 1.1369 ≈ 39.52 m

Answer for 1: The width of the river between B and C is approximately 39.52 m.

2. Juma wants to invest shs 150,000 at a rate of 10% compounded annually and accumulate the principal to shs 250,000. Using a calculator with a log key, find how long this will take, given that:

$S = P(1 + i)^n$

Where i = interest

n = number of years

P = Principal

Given:

S = 250,000

P = 150,000

i = 10% = 0.10

Formula: $S = P(1 + i)^n$

Substitute the values:

$250,000 = 150,000 \times (1 + 0.10)^n$

$250,000 / 150,000 = (1.10)^n$

$5/3 = (1.10)^n$

$1.6667 = (1.10)^n$

Take the natural logarithm on both sides:

$\ln(1.6667) = \ln((1.10)^n)$

$\ln(1.6667) = n \times \ln(1.10)$

Calculate the logarithms:

$\ln(1.6667) \approx 0.5108$

$\ln(1.10) \approx 0.0953$

$n = 0.5108 / 0.0953 \approx 5.36$ years

Answer for 2: It will take approximately 5.36 years.

3. (a) Find an equation (in the form Ax + By + C = 0) of the line which passes through the point (2, -1) and through the point of intersection of the line

$3x + y + 7 = 0$ and $10x - y + 38 = 0$

First, find the point of intersection of the lines $3x + y + 7 = 0$ and $10x - y + 38 = 0$:

Add the equations:

$(3x + y + 7) + (10x - y + 38) = 0$

$13x + 45 = 0$

$13x = -45$

$x = -45/13$

Substitute $x = -45/13$ into $3x + y + 7 = 0$:

$3(-45/13) + y + 7 = 0$

$-135/13 + y + 7 = 0$

$y = 135/13 - 7 = 135/13 - 91/13 = 44/13$

Point of intersection is (-45/13, 44/13).

Now find the equation of the line passing through (2, -1) and (-45/13, 44/13):

Slope $m = (y_2 - y_1) / (x_2 - x_1) = (44/13 - (-1)) / (-45/13 - 2) = (44/13 + 13/13) / (-45/13 - 26/13) = (57/13) / (-71/13) = 57 / -71 = -57/71$

Use point-slope form with point (2, -1):

$y - (-1) = (-57/71)(x - 2)$

$y + 1 = (-57/71)x + (57/71) \times 2$

$y + 1 = (-57/71)x + 114/71$

$y = (-57/71)x + 114/71 - 1$

$y = (-57/71)x + (114/71 - 71/71) = (-57/71)x + 43/71$

Convert to $Ax + By + C = 0$:

$(57/71)x + y - 43/71 = 0$

Multiply through by 71:

$57x + 71y - 43 = 0$

Answer for 3(a): The equation of the line is $57x + 71y - 43 = 0$.

3. (b) Find the equation of the perpendicular bisector of the line joining the points A(2, -3) and B(6, 5).

Midpoint of AB:

$((2 + 6)/2, (-3 + 5)/2) = (4, 1)$

Slope of AB:

$m_{AB} = (5 - (-3)) / (6 - 2) = 8 / 4 = 2$

Slope of the perpendicular bisector:

$m_{perpendicular} = -1 / m_{AB} = -1 / 2$

Equation of the perpendicular bisector passing through (4, 1):

$y - 1 = (-1/2)(x - 4)$

$y - 1 = (-1/2)x + 2$

$y = (-1/2)x + 3$

Convert to $Ax + By + C = 0$:

$(1/2)x + y - 3 = 0$

Multiply through by 2:

$x + 2y - 6 = 0$

Answer for 3(b): The equation of the perpendicular bisector is $x + 2y - 6 = 0$.

4. (a) Using the same xy plane, sketch the graphs of f: x --> 5 - x and g: x --> x and hence calculate the area of the triangle enclosed by the two graphs and the x axis

Graph of $f(x) = 5 - x$:

When $x = 0$, $y = 5$ (point: (0, 5))

When $y = 0$, $0 = 5 - x \Rightarrow x = 5$ (point: (5, 0))

Graph of $g(x) = x$:

Passes through (0, 0), (1, 1), etc.

Find intersection of f(x) and g(x):

$5 - x = x$

$5 = 2x$

$x = 2.5$

$y = 2.5$

Intersection point: (2.5, 2.5)

The triangle is formed by points (0, 0), (2.5, 2.5), and (5, 0).

Base of the triangle (along x-axis): $5 - 0 = 5$

Height of the triangle (at $x = 2.5$): $y = 2.5$

Area of the triangle = $(1/2) \times \text{base} \times \text{height} = (1/2) \times 5 \times 2.5 = 6.25$

Answer for 4(a): The area of the triangle is 6.25 square units.

4. (b) (i) Find $f^{-1}(x)$ if f(x) = e^x

Let y = f(x) = e^x

Swap x and y: x = e^y

Solve for y: y = ln(x)

So, $f^{-1}(x)$ = ln(x)

Answer for 4(b)(i): $f^{-1}(x)$ = ln(x)

4. (b) (ii) Sketch the graph of y = e^x and its inverse using the same xy plane

Graph of y = e^x:

At x = 0, y = e^0 = 1 (point: (0, 1))

At x = 1, y = e ≈ 2.718

Exponential growth, approaches y = 0 as x --> -∞.

Graph of y = ln(x) (the inverse):

At x = 1, y = ln(1) = 0 (point: (1, 0))

At x = e, y = ln(e) = 1

Logarithmic growth, defined for x > 0.

The graphs are reflections of each other over the line y = x.

Answer for 4(b)(ii): The graphs of y = e^x and y = ln(x) are sketched, intersecting at (0, 1) and (1, 0), symmetric about y = x.

4. (b) (iii) What is your conclusion about the value of log_e e^0?

log_e e^0 = log_e 1 = 0 (since e^0 = 1 and ln(1) = 0).

Answer for 4(b)(iii): The value of log_e e^0 is 0.

5. (a) The sum of three consecutive numbers in an arithmetic progression (AP) is 33. The product of the three numbers is 1323. Find the values of the numbers.

Let the three consecutive numbers in AP be a - d, a, a + d.

Sum: (a - d) + a + (a + d) = 3a = 33

$a = 11$

Product: $(a - d)(a)(a + d) = a(a^2 - d^2) = 1323$

$11(11^2 - d^2) = 1323$

$11(121 - d^2) = 1323$

$121 - d^2 = 1323 / 11 = 120.2727$

This leads to a non-integer d, so let's try three consecutive numbers directly:

Let the numbers be $n - 1$, n, $n + 1$ (common difference $d = 1$):

Sum: $(n - 1) + n + (n + 1) = 3n = 33$

$n = 11$

Numbers: 10, 11, 12

Product: $10 \times 11 \times 12 = 1320$ (close to 1323, possible typo in problem).

If the product is exactly 1323, adjust d:

$(11 - d)(11)(11 + d) = 1323$

$11(121 - d^2) = 1323$

$121 - d^2 = 120.2727$

$d^2 = 0.7273$ (not an integer, so likely a typo).

Assuming the product is 1320:

Numbers are 10, 11, 12.

Answer for 5(a): The numbers are 10, 11, 12 (assuming product is 1320 due to inconsistency).

5. (b) Given that the first term of a geometric progression (GP) is 1 and its common ratio is 1/2, find the:

(i) 7th term of the G.P.

(ii) sum to infinity of the series.

(i) 7th term of the GP:

Formula: $a_n = a \times r^{(n-1)}$

$a = 1$, $r = 1/2$, $n = 7$

$a_7 = 1 \times (1/2)^{(7-1)} = (1/2)^6 = 1/64$

Answer for 5(b)(i): The 7th term is 1/64.

(ii) Sum to infinity:

Formula: $S_\infty = a / (1 - r)$

$a = 1, r = 1/2$

$S_\infty = 1 / (1 - 1/2) = 1 / (1/2) = 2$

Answer for 5(b)(ii): The sum to infinity is 2.

Final Answers:

The width of the river is approximately 39.52 m.

It will take approximately 5.36 years.

6. (a) y is directly proportional to z^2 and is inversely proportional to x. If y = 3 when x = 2 and z = 1;

(i) Deduce the equation connecting x, y and z, then evaluate y when x = 10 and z = 5.

Since y is directly proportional to z^2 and inversely proportional to x:

$y = k \times (z^2 / x)$

Given y = 3, x = 2, z = 1:

$3 = k \times (1^2 / 2)$

$3 = k \times (1/2)$

$k = 3 \times 2 = 6$

The equation is:

$y = 6 \times (z^2 / x)$

Now evaluate y when x = 10, z = 5:

$y = 6 \times (5^2 / 10) = 6 \times (25 / 10) = 6 \times 2.5 = 15$

Answer for 6(a)(i): The equation is $y = 6 \times (z^2 / x)$, and y = 15 when x = 10 and z = 5.

(ii) Write y as a function of x when z = 20.

From the equation $y = 6 \times (z^2 / x)$, substitute z = 20:

$y = 6 \times (20^2 / x) = 6 \times (400 / x) = 2400 / x$

Answer for 6(a)(ii): $y = 2400 / x$

6. (b) The following table, (table 1) shows corresponding values of two variables F and C. Plot the values on a graph, hence estimate the relationship between the variables.

C | 10 | 20 | 30 | 40 | 50 | 60

F | 50 | 68 | 86 | 104 | 122 | 140

To estimate the relationship, calculate the slope between points to check if F and C have a linear relationship:

Slope between (10, 50) and (20, 68):

$(68 - 50) / (20 - 10) = 18 / 10 = 1.8$

Slope between (20, 68) and (30, 86):

$(86 - 68) / (30 - 20) = 18 / 10 = 1.8$

Slope between (30, 86) and (40, 104):

$(104 - 86) / (40 - 30) = 18 / 10 = 1.8$

The slope is constant (1.8), so the relationship is linear: $F = m \times C + b$.

Using slope $m = 1.8$ and point (10, 50):

$50 = 1.8 \times 10 + b$

$50 = 18 + b$

$b = 32$

The relationship is:

$F = 1.8 \times C + 32$

Answer for 6(b): The relationship is $F = 1.8 \times C + 32$ (linear). Plotting would show a straight line.

7. Given that $2A + B = 45°$, show that:

$\tan B = (1 - \tan^2 A) / (1 - \tan^2 A + 2\tan A)$

Hence find the value of $\tan(-15°)$ without using a calculator or mathematical tables. Simplify your answer, and rationalize the denominator.

First, solve for B:

$2A + B = 45°$

$B = 45° - 2A$

Use the tangent identity: $\tan(45° - 2A) = (\tan 45° - \tan 2A) / (1 + \tan 45° \times \tan 2A)$

$\tan 45° = 1$, so:

$\tan B = (1 - \tan 2A) / (1 + \tan 2A)$

Use the double-angle identity: $\tan 2A = 2\tan A / (1 - \tan^2 A)$:

$\tan B = (1 - (2\tan A / (1 - \tan^2 A))) / (1 + (2\tan A / (1 - \tan^2 A)))$

Simplify:

Numerator: $1 - (2\tan A / (1 - \tan^2 A)) = (1 - \tan^2 A - 2\tan A) / (1 - \tan^2 A)$

Denominator: $1 + (2\tan A / (1 - \tan^2 A)) = (1 - \tan^2 A + 2\tan A) / (1 - \tan^2 A)$

$\tan B = [(1 - \tan^2 A - 2\tan A) / (1 - \tan^2 A)] / [(1 - \tan^2 A + 2\tan A) / (1 - \tan^2 A)]$

$\tan B = (1 - \tan^2 A - 2\tan A) / (1 - \tan^2 A + 2\tan A)$

Factor out -1 from the numerator:

$\tan B = (- (\tan^2 A + 2\tan A - 1)) / (1 - \tan^2 A + 2\tan A)$

Notice the denominator can be rewritten:

Let $u = \tan A$, then:

$\tan B = (1 - u^2 - 2u) / (1 - u^2 + 2u)$

Now find $\tan(-15°)$:

If $B = -15°$, then:

$2A + (-15°) = 45°$

$2A = 60°$

$A = 30°$

$\tan A = \tan 30° = 1/\sqrt{3}$

$\tan^2 A = (1/\sqrt{3})^2 = 1/3$

Substitute into the equation:

$\tan(-15°) = (1 - (1/3) - 2(1/\sqrt{3})) / (1 - (1/3) + 2(1/\sqrt{3}))$

Numerator: $1 - 1/3 - 2/\sqrt{3} = 2/3 - 2/\sqrt{3}$

Denominator: $1 - 1/3 + 2/\sqrt{3} = 2/3 + 2/\sqrt{3}$

$\tan(-15°) = (2/3 - 2/\sqrt{3}) / (2/3 + 2/\sqrt{3})$

Multiply numerator and denominator by 3 to clear fractions:

Numerator: $2 - 2\sqrt{3}$

Denominator: $2 + 2\sqrt{3}$

$\tan(-15°) = (2 - 2\sqrt{3}) / (2 + 2\sqrt{3})$

Rationalize the denominator:

Multiply by conjugate $(2 - 2\sqrt{3})$:

Numerator: $(2 - 2\sqrt{3})(2 - 2\sqrt{3}) = (2 - 2\sqrt{3})^2 = 4 - 8\sqrt{3} + 12 = 16 - 8\sqrt{3}$

Denominator: $(2 + 2\sqrt{3})(2 - 2\sqrt{3}) = 4 - (2\sqrt{3})^2 = 4 - 12 = -8$

$\tan(-15°) = (16 - 8\sqrt{3}) / (-8) = - (16 - 8\sqrt{3}) / 8 = - (2 - \sqrt{3}) = \sqrt{3} - 2$

Answer for 7: $\tan(-15°) = \sqrt{3} - 2$

8. (a) Solve for the real number x if

$\log_{10} x = \log_5 2x$.

Rewrite $\log_5 2x$ using the change of base formula:

$\log_5 2x = (\log_{10} 2x) / (\log_{10} 5)$

The equation becomes:

$\log_{10} x = (\log_{10} 2x) / (\log_{10} 5)$

Let $\log_{10} 5 = k$, and let $y = \log_{10} x$, so $\log_{10} 2x = \log_{10} 2 + \log_{10} x = \log_{10} 2 + y$:

$y = (y + \log_{10} 2) / k$

$y \times k = y + \log_{10} 2$

$y \times k - y = \log_{10} 2$

$y (k - 1) = \log_{10} 2$

$y = (\log_{10} 2) / (k - 1)$

$\log_{10} x = (\log_{10} 2) / (\log_{10} 5 - 1)$

Numerically: $\log_{10} 2 \approx 0.3010$, $\log_{10} 5 \approx 0.6990$, so:

$\log_{10} x = 0.3010 / (0.6990 - 1) = 0.3010 / (-0.3010) = -1$

$x = 10^{(-1)} = 0.1$

Answer for 8(a): $x = 0.1$

8. (b) Simplify

$(1 / (y^2 + x^2)) + (1 / (y^2 - x^2))$

Find a common denominator:

$(y^2 + x^2)(y^2 - x^2)$

Rewrite:

$((y^2 - x^2) + (y^2 + x^2)) / ((y^2 + x^2)(y^2 - x^2)) = (2y^2) / (y^4 - x^4)$

Answer for 8(b): $(2y^2) / (y^4 - x^4)$

9. (a) Given that $f(x) = x^2 - x + 3$, find the value of $f'(x)$ from first principles.

$f'(x) = \lim(h \to 0) [(f(x + h) - f(x)) / h]$

$f(x + h) = (x + h)^2 - (x + h) + 3 = x^2 + 2xh + h^2 - x - h + 3$

$f(x) = x^2 - x + 3$

$f(x + h) - f(x) = (x^2 + 2xh + h^2 - x - h + 3) - (x^2 - x + 3) = 2xh + h^2 - h$

$(f(x + h) - f(x)) / h = (2xh + h^2 - h) / h = 2x + h - 1$

Take the limit as $h \to 0$:

$f'(x) = 2x - 1$

Answer for 9(a): $f'(x) = 2x - 1$

9. (b) Given that $y = \ln((3x - 2) / (x + 1))$, find dy/dx.

Use the quotient rule for logarithms:

$y = \ln(3x - 2) - \ln(x + 1)$

Differentiate:

$dy/dx = (d/dx)[\ln(3x - 2)] - (d/dx)[\ln(x + 1)]$

$= (3 / (3x - 2)) - (1 / (x + 1))$

Combine over a common denominator:

$= [(3(x + 1) - (3x - 2)) / ((3x - 2)(x + 1))] = (3x + 3 - 3x + 2) / ((3x - 2)(x + 1)) = 5 / ((3x - 2)(x + 1))$

Answer for 9(b): $dy/dx = 5 / ((3x - 2)(x + 1))$

10. (a) A curve that passes through the origin has a gradient 2x - 1. Find the equation of this curve in terms of x and y.

The gradient is $dy/dx = 2x - 1$.

Integrate:

$y = \int (2x - 1)\, dx = x^2 - x + C$

The curve passes through (0, 0):

$0 = 0^2 - 0 + C$

$C = 0$

The equation is:

$y = x^2 - x$

Answer for 10(a): $y = x^2 - x$

10. (b) $\int \sec^2 x\, dx$ by substituting $\tan x = t$.

Let $t = \tan x$, then $dt/dx = \sec^2 x$, so $dt = \sec^2 x\, dx$.

The integral becomes:

$\int \sec^2 x\, dx = \int dt = t + C = \tan x + C$

Answer for 10(b): $\int \sec^2 x\, dx = \tan x + C$

11. (a) Given the points A(2, -1) and B(3, 3), find:

(i) A vector from point A to point B in terms of the unit vectors i and j.

The position vector of A is 2i - j, and B is 3i + 3j.

Vector AB = OB - OA = (3i + 3j) - (2i - j) = (3 - 2)i + (3 - (-1))j = i + 4j

Answer for 11(a)(i): AB = i + 4j

(ii) The length of the vector AB.

AB = i + 4j

Length of AB = $\sqrt{(1^2 + 4^2)} = \sqrt{(1 + 16)} = \sqrt{17}$

Answer for 11(a)(ii): Length of AB = $\sqrt{17}$

(iii) The unit vector in the direction of vector BA.

First, find BA:

BA = -AB = -(i + 4j) = -i - 4j

Length of BA = $\sqrt{((-1)^2 + (-4)^2)} = \sqrt{(1 + 16)} = \sqrt{17}$

Unit vector in direction of BA = BA / |BA| = (-i - 4j) / $\sqrt{17}$

Answer for 11(a)(iii): Unit vector in direction of BA = (-i - 4j) / $\sqrt{17}$

11. (b) A line passes through the point (2, -1, 4) and is in the direction of the vector i + j - 2k. Find the:

(i) vector equation of the line.

The point (2, -1, 4) corresponds to position vector $r_0 = 2i - j + 4k$.

Direction vector d = i + j - 2k.

Vector equation of the line: $r = r_0 + t \times d$

r = (2i - j + 4k) + t(i + j - 2k)

Answer for 11(b)(i): r = (2i - j + 4k) + t(i + j - 2k)

(ii) angle the line makes with the positive x axis.

Direction vector of the line: d = i + j - 2k.

Direction vector of the positive x-axis: i = (1, 0, 0).

Use the dot product to find the angle θ:

$d \cdot i = (1 \times 1) + (1 \times 0) + (-2 \times 0) = 1$

$|d| = \sqrt{(1^2 + 1^2 + (-2)^2)} = \sqrt{(1 + 1 + 4)} = \sqrt{6}$

|i| = 1

$\cos \theta = (d \cdot i) / (|d| \times |i|) = 1 / (\sqrt{6} \times 1) = 1 / \sqrt{6}$

$\theta = \cos^{-1}(1 / \sqrt{6}) \approx \cos^{-1}(0.4082) \approx 65.9°$

Answer for 11(b)(ii): The angle with the positive x-axis is approximately 65.9°.

12. You are provided with the following frequency distribution table (table 2).

i | 1 | 2 | 3 | 4 | 5

x_i | 13 | 14 | 15 | 16 | 17

f_i | 1 | 4 | 12 | 2 | 1

(a) Find the value of

(i) $\Sigma(x_i \times f_i) / \Sigma f_i$

First, calculate $\Sigma(x_i \times f_i)$:

$(1 \times 13) + (2 \times 14) + (3 \times 15) + (4 \times 16) + (5 \times 17) = 13 + 28 + 45 + 64 + 85 = 235$

Calculate Σf_i:

$1 + 4 + 12 + 2 + 1 = 20$

Mean = $\Sigma(x_i \times f_i) / \Sigma f_i = 235 / 20 = 11.75$

Answer for 12(a)(i): $\Sigma(x_i \times f_i) / \Sigma f_i = 11.75$

(ii) $[\Sigma (x_i - X)^2 f_i] / \Sigma f_i$, where $X = [\Sigma (x_i \times f_i)] / \Sigma f_i$

From (i), X = 11.75.

Calculate $\Sigma (x_i - X)^2 f_i$:

For i = 1: $x_1 = 13$, $(13 - 11.75)^2 \times 1 = (1.25)^2 \times 1 = 1.5625$

For i = 2: $x_2 = 14$, $(14 - 11.75)^2 \times 4 = (2.25)^2 \times 4 = 5.0625 \times 4 = 20.25$

For i = 3: $x_3 = 15$, $(15 - 11.75)^2 \times 12 = (3.25)^2 \times 12 = 10.5625 \times 12 = 126.75$

For i = 4: $x_4 = 16$, $(16 - 11.75)^2 \times 2 = (4.25)^2 \times 2 = 18.0625 \times 2 = 36.125$

For i = 5: $x_5 = 17$, $(17 - 11.75)^2 \times 1 = (5.25)^2 \times 1 = 27.5625$

Sum: 1.5625 + 20.25 + 126.75 + 36.125 + 27.5625 = 212.25

$\Sigma f_i = 20$

Variance = $\Sigma (x_i - X)^2 f_i / \Sigma f_i = 212.25 / 20 = 10.6125$

Answer for 12(a)(ii): $[\Sigma (x_i - X)^2 f_i] / \Sigma f_i = 10.6125$

12. (b) Find the:

(i) median of the frequency distribution.

Total frequency $\Sigma f_i = 20$.

Median position = (20 + 1) / 2 = 10.5, so we take the average of the 10th and 11th values.

Cumulative frequencies:

i = 1: 1

i = 2: 1 + 4 = 5

i = 3: 5 + 12 = 17 (10th and 11th values fall here, $x_i = 15$)

Median = 15 (since both 10th and 11th values are 15).

Answer for 12(b)(i): Median = 15

(ii) mode of the frequency distribution.

Mode is the x_i with the highest frequency:

$f_i = 12$ at $x_i = 15$

Answer for 12(b)(ii): Mode = 15

13. (a) Two dice are thrown together. What is the probability of a score of an eight?

Possible outcomes for a score of 8 with two dice: (2, 6), (3, 5), (4, 4), (5, 3), (6, 2).

Number of favorable outcomes = 5.

Total possible outcomes with two dice = 6 × 6 = 36.

Probability = Number of favorable outcomes / Total outcomes = 5 / 36.

Answer for 13(a): The probability of a score of 8 is 5/36.

13. (b) Two of my friends and I, play a game of pure chance three times. What is the probability of me winning:

(i) every time?

Probability of winning each game (pure chance, 3 players) = 1/3.

Probability of winning all three times = $(1/3) \times (1/3) \times (1/3) = (1/3)^3 = 1/27$.

Answer for 13(b)(i): The probability of winning every time is 1/27.

(ii) only the third time?

Probability of losing the first game = 2/3.

Probability of losing the second game = 2/3.

Probability of winning the third game = 1/3.

Probability = $(2/3) \times (2/3) \times (1/3) = (2/3)^2 \times (1/3) = 4/9 \times 1/3 = 4/27$.

Answer for 13(b)(ii): The probability of winning only the third time is 4/27.

14. (a) If T is a linear transformation such that:

T = [a b]

[c d]

and T(x, y) = (3y, 5x),

Find the matrix T hence evaluate T(0, 0).

T(x, y) = (3y, 5x) means:

$a \times x + b \times y = 3y$

$c \times x + d \times y = 5x$

Equate coefficients:

For the first equation: $a \times x + b \times y = 3y$

$a \times x = 0 \Rightarrow a = 0$

$b \times y = 3y \Rightarrow b = 3$

For the second equation: $c \times x + d \times y = 5x$

$c \times x = 5x \Rightarrow c = 5$

$d \times y = 0 \Rightarrow d = 0$

Matrix T = [0 3]

[5 0]

Evaluate T(0, 0):

T(0, 0) = [0 3] [0] = [0 × 0 + 3 × 0] = [0]

[5 0] [0]   [5 × 0 + 0 × 0]   [0]

Answer for 14(a): Matrix T = [0 3]

[5 0], and T(0, 0) = (0, 0)

14. (b) Use the inverse matrix method to solve:

2y + 3x - 15 = 0

2x - 20 + 3y = 0

Rewrite in standard form:

3x + 2y = 15

2x + 3y = 20

Matrix form: AX = B

A = [3 2]

[2 3], X = [x], B = [15]

[y] [20]

Find the inverse of A:

Determinant of A = (3 × 3) - (2 × 2) = 9 - 4 = 5

Adjugate of A = [3 -2]

[-2 3]

A^{-1} = (1/5) × [3 -2]

[-2 3]

Solve X = A^{-1} × B:

[x] = (1/5) × [3 -2] [15] = (1/5) × [(3 × 15) + (-2 × 20)] = (1/5) × [45 - 40] = (1/5) × [5] = [1]

[y] [-2 3] [20] [(-2 × 15) + (3 × 20)] [-30 + 60] [30] [6]

So, x = 1, y = 6.

Answer for 14(b): x = 1, y = 6

15. A certain farmer wants to use part of his shamba to plant cabbages and potatoes. He divides the shamba into several equal sized portions. The farming of cabbages will cost shs. 24,000 per portion and potatoes shs. 8,000 per portion. The minimum funds which can be used in farming the two crops is shs. 240,000. Cabbages require 10 man hours per portion, while potatoes require 200 man hours per portion. The estimated profits is shs. 16,000 per portion of cabbages and shs. 12,000 per portion of potatoes.

(a) How should he allocate the expected shamba for maximum profit?

Let x be the number of portions for cabbages, y for potatoes.

Constraints:

Cost: $24000x + 8000y \geq 240000$

$3x + y \geq 30$

Man hours: $10x + 200y$ (no upper limit given, so we focus on cost).

$x \geq 0, y \geq 0$

Objective: Maximize profit $P = 16000x + 12000y$

Solve the cost constraint as equality: $3x + y = 30$

$y = 30 - 3x$

Vertices of the feasible region:

(0, 30): $y = 30, x = 0$

(10, 0): $3x = 30, x = 10, y = 0$

Evaluate P at vertices:

(0, 30): $P = 16000 \times 0 + 12000 \times 30 = 360000$

(10, 0): $P = 16000 \times 10 + 12000 \times 0 = 160000$

Maximum profit occurs at (0, 30), so allocate all portions to potatoes.

Answer for 15(a): Allocate all portions to potatoes (0 portions for cabbages, 30 for potatoes).

(b) What is the maximum profit?

From (a), at (0, 30):

$P = 16000 \times 0 + 12000 \times 30 = 360000$

Answer for 15(b): Maximum profit = shs. 360,000

16. (a) At time t, the position vectors of two particles P and Q are given by

$P = 2t\, i + (3t^2 - 4t)\, j - t\, k$

$Q = t\, i - 2t\, j + (2t^2 - 1)\, k$

Find the velocity and acceleration of Q relative to P when t = 1.

Position vector of P: $P = 2t\, i + (3t^2 - 4t)\, j - t\, k$

Position vector of Q: $Q = t\, i - 2t\, j + (2t^2 - 1)\, k$

Velocity of P = $dP/dt = 2\, i + (6t - 4)\, j - k$

At t = 1: $v_P = 2\, i + (6 \times 1 - 4)\, j - k = 2\, i + 2\, j - k$

Velocity of Q = $dQ/dt = 1\, i - 2\, j + 4t\, k$

At t = 1: $v_Q = 1\, i - 2\, j + 4 \times 1\, k = i - 2\, j + 4\, k$

Velocity of Q relative to P: $v_Q - v_P = (i - 2\, j + 4\, k) - (2\, i + 2\, j - k) = (1 - 2)\, i + (-2 - 2)\, j + (4 - (-1))\, k = -i - 4\, j + 5\, k$

Acceleration of P = $d^2P/dt^2 = d(v_P)/dt = 0\, i + 6\, j + 0\, k$

At t = 1: $a_P = 6\, j$

Acceleration of Q = $d^2Q/dt^2 = d(v_Q)/dt = 0\, i + 0\, j + 4\, k$

At t = 1: $a_Q = 4\, k$

Acceleration of Q relative to P: $a_Q - a_P = 4\, k - 6\, j = -6\, j + 4\, k$

Answer for 16(a): Velocity of Q relative to P at t = 1 is $-i - 4j + 5k$, acceleration is $-6j + 4k$.

16. (b) A particle of unit mass moves so that its position vector r at time t seconds is given by

$r = (\cos t)\, i + (\sin t)\, j + \frac{1}{2} t^2\, k$

Find the:

(i) momentum at time t.

Mass = 1.

Velocity $v = dr/dt = -\sin t\, i + \cos t\, j + t\, k$

Momentum = mass × velocity = $1 \times v = -\sin t\, i + \cos t\, j + t\, k$

Answer for 16(b)(i): Momentum = $-\sin t\, i + \cos t\, j + t\, k$

(ii) kinetic energy at time t.

Kinetic energy = $(1/2) \times \text{mass} \times |v|^2$

$|v|^2 = (-\sin t)^2 + (\cos t)^2 + (t)^2 = \sin^2 t + \cos^2 t + t^2 = 1 + t^2$

Kinetic energy = $(1/2) \times 1 \times (1 + t^2) = (1 + t^2) / 2$

Answer for 16(b)(ii): Kinetic energy = $(1 + t^2) / 2$

(iii) force acting on the particle at time t.

Force = mass × acceleration

Acceleration a = dv/dt = -cos t i - sin t j + 1 k

Force = 1 × a = -cos t i - sin t j + k

Answer for 16(b)(iii): Force = -cos t i - sin t j + k

(iv) power exerted by the force in 16(b)(iii) above at time t.

Power = force · velocity

F = -cos t i - sin t j + k

v = -sin t i + cos t j + t k

Power = (-cos t)(-sin t) + (-sin t)(cos t) + (1)(t) = cos t sin t - sin t cos t + t = 0 + t = t

Answer for 16(b)(iv): Power = t