THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

141 BASIC APPLIED MATHEMATICS

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2010**

Instructions

1. This paper consists of **Ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

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1. The coordinates for points A, B and C are (2, 9), (4, 3) and (2, -5) respectively. If the line through C with gradient -1/2 meets the line AB produced at D, find:

(a) the coordinates of D

Line AB: Slope = (3 - 9)/(4 - 2) = -6/2 = -3

Equation of AB: y - 9 = -3(x - 2) → y = -3x + 15

Line through C(2, -5) with slope -1/2: y - (-5) = (-1/2)(x - 2) → y + 5 = (-1/2)x + 1 → y = (-1/2)x - 4

Intersection at D: -3x + 15 = (-1/2)x - 4

Multiply by 2: -6x + 30 = -x - 8 → -5x = -38 → x = 38/5

y = -3(38/5) + 15 = -114/5 + 75/5 = -39/5

D = (38/5, -39/5)

Answer: D = (38/5, -39/5)

(b) the equation of the line through D perpendicular to the line 5y - 4x = 17

Line 5y - 4x = 17 → y = (4/5)x + 17/5 → slope = 4/5

Perpendicular slope: -5/4

Line through D(38/5, -39/5): y - (-39/5) = (-5/4)(x - 38/5)

y + 39/5 = (-5/4)x + 190/20 → y = (-5/4)x + 95/10 - 78/10 = (-5/4)x + 17/10

Multiply by 4: 4y = -5x + 34/5 → 20y + 25x - 34 = 0

Answer: 25x + 20y - 34 = 0

2. A function is defined by f(x) = 1/(1 - x), x ≠ 1

(a) Why is 1 excluded from the domain of f

f(x) = 1/(1 - x) → when x = 1, denominator = 0, which is undefined.

Answer: 1 is excluded because f(1) = 1/(1 - 1) is undefined (division by zero).

(b) Sketch the curve y = f(x)

y = 1/(1 - x)

Vertical asymptote: x = 1 (denominator = 0)

Horizontal asymptote: As $x \to \pm\infty$, $y \to 0$

Points: $x = 0 \to y = 1$; $x = 2 \to y = -1$; $x = -1 \to y = 1/2$

Behavior: As $x \to 1^-$, $y \to +\infty$; as $x \to 1^+$, $y \to -\infty$

Answer: Curve has a vertical asymptote at $x = 1$, horizontal asymptote at $y = 0$, passes through (0, 1), (2, -1), (-1, 1/2). (I can confirm if you'd like to generate a sketch.)

(c) Find $f^{-1}(x)$ in terms of x and give the domain and range of f^{-1}

$y = 1/(1 - x) \to x = 1/(1 - y) \to 1 - y = 1/x \to y = 1 - 1/x = (x - 1)/x$

$f^{-1}(x) = (x - 1)/x$

Domain of f^{-1}: $x \neq 0$ (since $f(x) \neq 0$)

Range of f^{-1}: $y \neq 1$ (since f^{-1} maps to the domain of f)

Answer: $f^{-1}(x) = (x - 1)/x$, Domain: $x \neq 0$, Range: $y \neq 1$

3. (a) Which term of the sequence 14, 21, 28,… is 168?

Arithmetic sequence: $a = 14$, $d = 7$

nth term: $a_n = a + (n-1)d = 14 + (n-1)7 = 7n + 7$

$7n + 7 = 168 \to 7n = 161 \to n = 23$

Answer: 23rd term

(b) In a certain geometric progression, the third term exceeds the first by 9 while the second term exceeds the fourth by 18. Find the numbers

Geometric sequence: a, ar, ar^2, ar^3

Third exceeds first by 9: $ar^2 - a = 9 \to a(r^2 - 1) = 9$

Second exceeds fourth by 18: $ar - ar^3 = 18 \to ar(1 - r^2) = 18$

Divide: $[a(r^2 - 1)] / [ar(1 - r^2)] = 9/18 \to (r^2 - 1) / [r(1 - r^2)] = 1/2$

$(r^2 - 1) = (1/2)r(1 - r^2) \to 2(r^2 - 1) = r - r^3 \to r^3 - 2r^2 - r + 2 = 0$

Solve: $r = 2$ (by inspection; $r^3 - 2r^2 - r + 2 = 8 - 8 - 2 + 2 = 0$)

$a(r^2 - 1) = 9 \to a(4 - 1) = 9 \to 3a = 9 \to a = 3$

Sequence: 3, 6, 12, 24

Answer: 3, 6, 12, 24

4. (a) If y varies jointly and directly as cube root of x and the square root of z, express this statement as an equation given that y = 2 when x = 8 and z = 1/4

$y \propto (x^{(1/3)})(z^{(1/2)}) \rightarrow y = k (x^{(1/3)})(z^{(1/2)})$

At x = 8, z = 1/4: y = 2

$x^{(1/3)} = 8^{(1/3)} = 2$, $z^{(1/2)} = (1/4)^{(1/2)} = 1/2$

$2 = k (2)(1/2) \rightarrow 2 = k \rightarrow k = 2$

Equation: $y = 2 (x^{(1/3)})(z^{(1/2)})$

Answer: $y = 2 (x^{(1/3)})(z^{(1/2)})$

(b) A bank uses the formula $A = P(1 + r/100)^n$ to calculate the amount of money in an account. Calculate A when P = 800, r = 6 and n = 5 correct to 2 decimal places

$A = 800 (1 + 6/100)^5 = 800 (1.06)^5$

$(1.06)^5 \approx 1.338225$

$A = 800 \times 1.338225 \approx 1070.58$

Answer: $A \approx 1070.58$

5. (a) Given that tan 75° = 2 + √3, find in the form m + n√3, where m and n are integers, the value of (i) tan 15° (ii) tan 105°

(i) tan 15°: tan 75° = tan (90° - 15°) = cot 15° = 1/tan 15°

1/tan 15° = 2 + √3 → tan 15° = 1/(2 + √3)

Rationalize: tan 15° = (2 - √3) / (4 - 3) = 2 - √3

(ii) tan 105°: tan 105° = tan (180° - 75°) = -tan 75° = -(2 + √3) = -2 - √3

Answer: (i) tan 15° = 2 - √3, (ii) tan 105° = -2 - √3

(b) Find, in radians to two decimal places, the values of x in the interval $0 \leq x \leq 2\pi$, for which $3 \sin^2 x + \sin x - 2 = 0$

Let $u = \sin x$: $3u^2 + u - 2 = 0$

Solve: $u = [-1 \pm \sqrt{(1 + 24)}]/6 = (-1 \pm 5)/6 \rightarrow u = 2/3$ or $u = -1$

$\sin x = 2/3$ (since $-1 \leq \sin x \leq 1$)

$x = \sin^{-1}(2/3) \approx 0.73$, or $x = \pi - 0.73 = 2.41$ (in $[0, 2\pi]$)

Answer: $x \approx 0.73, 2.41$

6. (a) Use the laws of logarithms to express $3 \ln 4 - \ln 1/8 + 1/2 \ln 2.25$ as a single logarithm in its simplest form, showing all your working

$3 \ln 4 = \ln (4^3) = \ln 64$

$\ln 1/8 = \ln (8^{-1}) = -\ln 8$

$1/2 \ln 2.25 = 1/2 \ln (9/4) = \ln (9/4)^{(1/2)} = \ln (3/2)$

Expression: $\ln 64 - (-\ln 8) + \ln (3/2) = \ln 64 + \ln 8 + \ln (3/2)$

$= \ln (64 \times 8 \times 3/2) = \ln (512 \times 3/2) = \ln 768$

Answer: $\ln 768$

(b) Given that $p = e^x$ and $q = e^y$, express the following involving either logarithms or powers of e:

(i) $e^{(x + y)}$

$e^{(x + y)} = e^x e^y = p q$

Answer: (i) $p q$

(ii) $e^{(2x - y)}$

$e^{(2x - y)} = e^{(2x)}/e^y = (e^x)^2/e^y = p^2/q$

Answer: (ii) p^2/q

7. (a) Find f'(x) from first principle, given that $f(x) = \sqrt{(2x + 1)}$

$f(x) = (2x + 1)^{(1/2)}$

$f'(x) = \lim(h\to 0)\ [(2(x + h) + 1)^{(1/2)} - (2x + 1)^{(1/2)}]/h$

Multiply by conjugate: $[(2x + 2h + 1) - (2x + 1)] / [h ((2x + 2h + 1)^{(1/2)} + (2x + 1)^{(1/2)})]$

$= 2h / [h ((2x + 2h + 1)^{(1/2)} + (2x + 1)^{(1/2)})] = 2 / [(2x + 2h + 1)^{(1/2)} + (2x + 1)^{(1/2)}]$

As $h \to 0$: $2 / (2(2x + 1)^{(1/2)}) = 1 / (2x + 1)^{(1/2)}$

Answer: $f'(x) = 1 / (2x + 1)^{(1/2)}$

(b) If $T = 2\pi \sqrt{(L/g)}$ where π and g are constants, find dL/dT

$T = 2\pi (L/g)^{(1/2)} = 2\pi/\sqrt{g}\ L^{(1/2)}$

$dT/dL = (2\pi/\sqrt{g})\ (1/2)\ L^{(-1/2)} = \pi/(\sqrt{g}\ \sqrt{L})$

$dL/dT = 1/(dT/dL) = (\sqrt{g}\ \sqrt{L})/\pi$

Answer: $dL/dT = (\sqrt{g}\ \sqrt{L})/\pi$

(c) A curve is represented parametrically by the equations $x = 1/t$, $y = 1/(1 + t)$. Find dy/dx in terms of t

$dx/dt = d/dt\ (1/t) = -1/t^2$

$dy/dt = d/dt\ [1/(1 + t)] = -1/(1 + t)^2$

$dy/dx = (dy/dt) / (dx/dt) = [-1/(1 + t)^2] / [-1/t^2] = t^2/(1 + t)^2$

Answer: $dy/dx = t^2/(1 + t)^2$

8. The graph shows sketches of the line $y = 3$ and the curve $y = x^2 - 3x + 5$ they intersect at the points A and B. The shaded region is bounded by the arc AB and the chord AB.

(a) Find the area of the shaded region

Curve: $y = x^2 - 3x + 5$, Line: $y = 3$

Intersection points A and B: $x^2 - 3x + 5 = 3 \to x^2 - 3x + 2 = 0 \to (x - 1)(x - 2) = 0 \to x = 1, x = 2$

Points: A(1, 3), B(2, 3)

Area between curve and line from $x = 1$ to $x = 2$: $\int$ (from 1 to 2) $[(x^2 - 3x + 5) - 3]\ dx$

$= \int$ (from 1 to 2) $(x^2 - 3x + 2)\ dx$

$= [x^3/3 - (3x^2)/2 + 2x]$ (from 1 to 2)

At $x = 2$: $(8/3) - (12/2) + 4 = 8/3 - 6 + 4 = 2/3$

At $x = 1$: $(1/3) - (3/2) + 2 = 1/3 - 3/2 + 2 = 5/6$

Area = $2/3 - 5/6 = 4/6 - 5/6 = -1/6$ (take absolute value since area is positive) = $1/6$

Answer: Area = $1/6$

(b) Show that the equation of the tangent to the curve at A is $y + x - 4 = 0$ and find the equation of the tangent to the curve at B

Curve: $y = x^2 - 3x + 5$

At A(1, 3): $dy/dx = 2x - 3 \rightarrow$ at $x = 1$, $dy/dx = 2(1) - 3 = -1$

Tangent at A: $y - 3 = -1(x - 1) \rightarrow y - 3 = -x + 1 \rightarrow y + x - 4 = 0$ (shown)

At B(2, 3): $dy/dx = 2(2) - 3 = 1$

Tangent at B: $y - 3 = 1(x - 2) \rightarrow y - 3 = x - 2 \rightarrow y - x + 1 = 0$

Answer: Tangent at A: $y + x - 4 = 0$ (shown); Tangent at B: $y - x + 1 = 0$

9. Three points P, Q and R have position vectors p, q and r respectively, where $p = 7i + 10j$, $q = 3i + 12j$, $r = -i + 4j$

(a) Write down the vectors PQ and RQ, show that they are perpendicular

PQ:

$PQ = Q - P = (3i + 12j) - (7i + 10j) = (3 - 7)i + (12 - 10)j = -4i + 2j$

RQ:

$RQ = Q - R = (3i + 12j) - (-i + 4j) = (3 - (-1))i + (12 - 4)j = 4i + 8j$

Check perpendicularity:

Vectors are perpendicular if their dot product is zero.

$PQ \cdot RQ = (-4)(4) + (2)(8) = -16 + 16 = 0$

Since the dot product is zero, PQ and RQ are perpendicular.

Answer: $PQ = -4i + 2j$, $RQ = 4i + 8j$; $PQ \cdot RQ = 0$, so they are perpendicular.

(b) If S is the midpoint of PR, show that $|QS| = |RS|$

Find S (midpoint of PR):

P = (7, 10), R = (-1, 4)

S = ((7 + (-1))/2, (10 + 4)/2) = (6/2, 14/2) = (3, 7)

QS:

QS = S - Q = (3i + 7j) - (3i + 12j) = (3 - 3)i + (7 - 12)j = 0i - 5j = -5j

RS:

RS = S - R = (3i + 7j) - (-i + 4j) = (3 - (-1))i + (7 - 4)j = 4i + 3j

Magnitudes:

$|QS| = \sqrt{(0^2 + (-5)^2)} = \sqrt{25} = 5$

$|RS| = \sqrt{(4^2 + 3^2)} = \sqrt{(16 + 9)} = \sqrt{25} = 5$

Since |QS| = |RS|, the condition is satisfied.

Answer: S = (3, 7), |QS| = 5, |RS| = 5; |QS| = |RS|, shown.

10. The marks obtained by 80 students in an examination are shown below.

Mark	Frequency
1-10	3
11-20	5
21-30	5
31-40	9
41-50	11
51-60	14
61-70	14
71-80	8
81-90	6
90-100	4

Plot a cumulative frequency curve and hence estimate:

(a) the median

Cumulative frequencies: 3, 8, 13, 22, 33, 47, 61, 69, 75, 79

Total frequency = 80, median at 40th position

40th position lies in 41-50 class (cumulative frequency 33 to 47)

Median = $L + [(n/2 - F)/f] \times w$

$L = 41$, $n/2 = 40$, $F = 33$, $f = 11$, $w = 10$

Median = $41 + [(40 - 33)/11] \times 10 = 41 + (7/11) \times 10 = 41 + 70/11 \approx 47.36$

Answer: Median ≈ 47.36

(b) the lower and the upper quartile

Q1 at 20th position (80/4): 20th in 31-40 class (13 to 22)

$Q1 = 31 + [(20 - 13)/9] \times 10 = 31 + (7/9) \times 10 \approx 38.78$

Q3 at 60th position (3×80/4): 60th in 51-60 class (47 to 61)

$Q3 = 51 + [(60 - 47)/14] \times 10 = 51 + (13/14) \times 10 \approx 60.29$

Answer: Q1 ≈ 38.78, Q3 ≈ 60.29

11. (a) A coin is tossed twice. List down the possible outcomes (sample space) and hence find the probability of obtaining at least one head

Sample space: {HH, HT, TH, TT}

At least one head: {HH, HT, TH} → 3 outcomes

Probability = 3/4

Answer: Sample space: {HH, HT, TH, TT}; Probability = 3/4

(b) Using the sample space (S) in part (a) above, determine the probability of a sample space is always one (1)

Probability of sample space S = {HH, HT, TH, TT} is P(S) = 1 (by definition).

The question may be misinterpreted; if it means probability of an event in S, we need clarification. Assuming it's asking to confirm P(S) = 1:

Answer: P(S) = 1

(c) Find the value of n that will satisfy the following equation $3 \times n!\, C_2 = 7 \times n!\, C_2$

$n!\ C_2 = n! / (2!(n-2)!) = n(n-1)/2$

Equation: $3 \times (n!\ C_2) = 7 \times (n!\ C_2)$

This implies $3k = 7k$ where $k = n!\ C_2$, which is only true if $k = 0$ (but $n!\ C_2 = 0$ only for $n < 2$, invalid).

$3 \times n\ C_2 = 7 \times (n-1)\ C_2$

$3 \times [n(n-1)/2] = 7 \times [(n-1)(n-2)/2]$

$3n(n-1) = 7(n-1)(n-2)$

$3n = 7(n-2) \rightarrow 3n = 7n - 14 \rightarrow 14 = 4n \rightarrow n = 7/2$

Try $3 \times n\ C_2 = 7 \times (n-2)\ C_2$: $3n(n-1) = 7(n-2)(n-3) \rightarrow 3n^2 - 3n = 7n^2 - 35n + 42 \rightarrow 4n^2 - 32n + 42 = 0 \rightarrow 2n^2 - 16n + 21 = 0$

$n = [16 \pm \sqrt{(256 - 168)}]/4 = (16 \pm \sqrt{88})/4 = (16 \pm 2\sqrt{22})/4 = 4 \pm \sqrt{22}/2$.

12. (a) Find $(A + B)^T$ given that A = [0 1] and B = [-2 1]

[2 -1]   [3 0]

A + B = [0 + (-2) 1 + 1] = [-2 2]

[2 + 3   -1 + 0]   [5 -1]

$(A + B)^T$ = [-2 5]

[2 -1]

Answer: $(A + B)^T$ = [-2 5]

[2 -1]

(b) Find the inverse of the matrix

C = [-1 1 -2]

[-2 2 1]

[1 -2 -3]

Det(C) = -1(2(-3) - 1(-2)) - 1(-2(-3) - 1(1)) + (-2)(-2(-2) - 2(1))

= -1(-6 + 2) - 1(6 - 1) + (-2)(4 - 2) = -1(-4) - 1(5) + (-2)(2) = 4 - 5 - 4 = -5

Adjoint of C:

$C_{11} = (2(-3) - 1(-2)) = -4$

$C_{12} = -(-2(-3) - 1(1)) = -(6 - 1) = -5$

$C_{13} = (-2(-2) - 2(1)) = 4 - 2 = 2$

$C_{21} = -((-1)(-3) - (-2)(-2)) = -(3 - 4) = 1$

$C_{22} = (-1(-3) - (-2)(1)) = 3 + 2 = 5$

$C_{23} = -((-1)(-2) - 2(1)) = -(2 - 2) = 0$

$C_{31} = (-1(1) - 2(-2)) = -1 + 4 = 3$

$C_{32} = -((-1)(1) - (-2)(-1)) = -(-1 - 2) = 3$

$C_{33} = (-1(2) - 2(-1)) = -2 + 2 = 0$

Adj(C) = [-4 1 3]

[-5 5 3]

[2   0 0]

C^{-1} = (1/-5) [-4 1 3] = [4/5 -1/5 -3/5]

[-5 5 3]   [1   -1   -3/5]

[2   0 0]   [-2/5 0   0]

Answer: C^{-1} = [4/5 -1/5 -3/5]

[1   -1   -3/5]

[-2/5 0   0]

(c) A restaurant sells three meals A, B and C. In two days the sales were as shown in matrix S.

Mon Tue

S = [10 5] A

[15 20] B

[10 10] C

The price in ($) paid for each meal is given by matrix P

A B C

P = [2 4 7]

Work out the matrix product PS and interpret the result

P is 1×3, S is 3×2 → PS is 1×2

PS = [2 4 7] [10 5] = [(2×10 + 4×15 + 7×10) (2×5 + 4×20 + 7×10)]

[15 20]

[10 10]

= [20 + 60 + 70 10 + 80 + 70] = [150 160]

Interpretation: 150 is the total revenue on Monday, 160 is the total revenue on Tuesday.

Answer: PS = [150 160]; 150 is Monday's revenue, 160 is Tuesday's revenue (in $).

13. Two mills produce the same three types of plywood. The table given below gives the production, demand and cost data.

Plywood type | Mill I per day | Mill II per day | Six – month demand

A | 100 sheets | 20 sheets | 2000 sheets

B | 40 sheets | 80 sheets | 3200 sheets

C | 60 sheets | 60 sheets | 3600 sheets

Daily costs | Tshs 3,000,000 | Tshs 2,000,000

Find the number of days that each mill should operate during the 6 months in order to supply the required sheets in the most economical way

Let x = days for Mill I, y = days for Mill II

Minimize cost: C = 3,000,000x + 2,000,000y

Constraints:

A: 100x + 20y ≥ 2000 → 5x + y ≥ 100

B: 40x + 80y ≥ 3200 → x + 2y ≥ 80

C: 60x + 60y ≥ 3600 → x + y ≥ 60

x, y ≥ 0

Vertices:

(0, 100): 5x + y = 100, x = 0

(16, 32): $5x + y = 100$, $x + 2y = 80 \rightarrow 5x + y - (x + 2y) = 20 \rightarrow 4x - y = 20 \rightarrow y = 4x - 20$, $5x + (4x - 20) = 100 \rightarrow 9x = 120 \rightarrow x = 40/3 \approx 13.33$, $y = 80 - 40/3 = 200/3 \approx 66.67$ (but use integer constraints for simplicity)

Adjust for exact: Intersection points:

$5x + y = 100$, $x + 2y = 80 \rightarrow 4x - y = 20 \rightarrow y = 4x - 20$, $x + 2(4x - 20) = 80 \rightarrow 9x - 40 = 80 \rightarrow 9x = 120 \rightarrow x = 40/3$, $y = 200/3$

$x + y = 60$, $x + 2y = 80 \rightarrow y = 20$, $x + 20 = 60 \rightarrow x = 40$

$5x + y = 100$, $x + y = 60 \rightarrow 4x = 40 \rightarrow x = 10$, $y = 50$

Vertices: (40, 20), (10, 50), (0, 80) (minimum for C)

Evaluate C:

(40, 20): $3{,}000{,}000(40) + 2{,}000{,}000(20) = 120{,}000{,}000 + 40{,}000{,}000 = 160{,}000{,}000$

(10, 50): $3{,}000{,}000(10) + 2{,}000{,}000(50) = 30{,}000{,}000 + 100{,}000{,}000 = 130{,}000{,}000$

(0, 80): $3{,}000{,}000(0) + 2{,}000{,}000(80) = 160{,}000{,}000$

Minimum at (10, 50).

Answer: Mill I: 10 days, Mill II: 50 days

14. (a) The gradient of a curve at the point (x, y) is $3x^2 - 4x + 1$. If the curve passes through the point (2, 3):

(i) Show that the curve also passes through the point A(-1, 3)

$dy/dx = 3x^2 - 4x + 1$

$y = \int (3x^2 - 4x + 1)\, dx = x^3 - 2x^2 + x + C$

At (2, 3): $3 = 2^3 - 2(2^2) + 2 + C \rightarrow 3 = 8 - 8 + 2 + C \rightarrow C = 1$

$y = x^3 - 2x^2 + x + 1$

At (-1, 3): $y = (-1)^3 - 2(-1)^2 + (-1) + 1 = -1 - 2 - 1 + 1 = -3 \neq 3$

The curve does not pass through (-1, 3). Let's try A(1, -3):

At (1, -3): $y = 1^3 - 2(1^2) + 1 + 1 = 1 - 2 + 1 + 1 = 1 \neq -3$

Possible typo: Let's find where y = 3: $3 = x^3 - 2x^2 + x + 1 \rightarrow x^3 - 2x^2 + x - 2 = 0$

x = 2 (known), solve for others: $(x - 2)(x^2 + 1) = 0 \rightarrow$ no other real roots. The problem may have an error.

Answer: The curve $y = x^3 - 2x^2 + x + 1$ does not pass through (-1, 3) (possible typo in problem).

(ii) Find the equation of the tangent at A(-1, 3)

Using the curve $y = x^3 - 2x^2 + x + 1$ (though it doesn't pass through (-1, 3), let's proceed as if corrected):

dy/dx at $x = -1$: $3(-1)^2 - 4(-1) + 1 = 3 + 4 + 1 = 8$

Tangent at (-1, 3): $y - 3 = 8(x - (-1)) \rightarrow y - 3 = 8x + 8 \rightarrow y = 8x + 11$

Answer: Tangent: $y = 8x + 11$ (assuming a correction in part (i)).

(iii) Find the maximum and minimum values of y

$dy/dx = 3x^2 - 4x + 1 = 0$

$x = [4 \pm \sqrt{(16 - 12)}]/6 = (4 \pm 2)/6 \rightarrow x = 1, x = 1/3$

$d^2y/dx^2 = 6x - 4$

At $x = 1$: $6(1) - 4 = 2 > 0 \rightarrow$ minimum

At $x = 1/3$: $6(1/3) - 4 = -2 < 0 \rightarrow$ maximum

y at $x = 1$: $1^3 - 2(1^2) + 1 + 1 = 1$

y at $x = 1/3$: $(1/3)^3 - 2(1/3)^2 + (1/3) + 1 = 1/27 - 2/9 + 1/3 + 1 = 4/27 + 1 = 31/27$

Answer: Maximum $y = 31/27$ at $x = 1/3$, minimum $y = 1$ at $x = 1$

(b) Use the definition $y + \Delta y = f(x + \Delta x)$, find the cube root of 1010 for $\Delta x = 10$ and $\Delta y = (dy/dx)\Delta x$

$f(x) = x^{(1/3)}$, $x = 1000$, $\Delta x = 10$

$dy/dx = (1/3)x^{(-2/3)}$

At $x = 1000$: $dy/dx = (1/3)(1000)^{(-2/3)} = (1/3)(1/100) = 1/300$

$\Delta y = (dy/dx)\Delta x = (1/300)(10) = 1/30$

$y = (1000)^{(1/3)} = 10$

$y + \Delta y = 10 + 1/30 = 301/30 \approx 10.033$

Answer: Cube root of 1010 ≈ 10.033

(c) Find:

(i) $\int \cos^2 x \, dx$

$\cos^2 x = (1 + \cos 2x)/2$

$\int \cos^2 x \, dx = \int (1 + \cos 2x)/2 \, dx = (1/2)(x + (\sin 2x)/2) + C = (x/2) + (\sin 2x)/4 + C$

Answer: (i) $(x/2) + (\sin 2x)/4 + C$

(ii) $\int$ (from 0 to 1) $(1/(x^2 + 2x + 5))\, dx$

$x^2 + 2x + 5 = (x + 1)^2 + 4$

$\int 1/((x + 1)^2 + 4)\, dx = (1/2) \tan^{-1}((x + 1)/2)$

From 0 to 1: $[(1/2) \tan^{-1}(2/2)] - [(1/2) \tan^{-1}(1/2)] = (1/2)(\pi/4 - \tan^{-1}(1/2))$

Answer: (ii) $(1/2)(\pi/4 - \tan^{-1}(1/2))$

15. (a) Sketch the curve $y = x + x^2$ for $-2 \le x \le 4$ and shade the area bounded by the curve, the x-axis and the lines $x = 2$ and $x = 3$

Generated Sketch Description:

Axes and Scale: Draw a Cartesian plane with x-axis from -2 to 4, y-axis from -2 to 20. Scale: 1 unit = 1 cm.

Curve: $y = x + x^2$

At $x = -2$: $y = -2 + 4 = 2$

At $x = 0$: $y = 0$

At $x = 2$: $y = 2 + 4 = 6$

At $x = 3$: $y = 3 + 9 = 12$

At $x = 4$: $y = 4 + 16 = 20$

Plot points (-2, 2), (0, 0), (2, 6), (3, 12), (4, 20); draw a parabola opening upward.

Shaded Area: Between $x = 2$ and $x = 3$, above the x-axis ($y \ge 0$), shade the region under the curve from (2, 6) to (3, 12).

Answer: Sketch: Parabola $y = x + x^2$ from $x = -2$ to 4, shaded area between $x = 2$ and $x = 3$ under the curve.

(b) Find the volume of the solid of revolution obtained by rotating the shaded area in part (a) above

Volume = $\pi \int$ (from 2 to 3) $(x + x^2)^2\, dx$

$(x + x^2)^2 = x^2 + 2x^3 + x^4$

$\int (x^2 + 2x^3 + x^4)\, dx = (x^3/3) + (x^4/2) + (x^5/5)$

From 2 to 3: $[(27/3) + (81/2) + (243/5)] - [(8/3) + (16/2) + (32/5)]$

$= [9 + 40.5 + 48.6] - [8/3 + 8 + 6.4] = 98.1 - 17.0667 = 81.0333$

Volume $= \pi \times 81.0333 \approx 254.573$

Answer: Volume ≈ 254.573

(c) Find the coordinates of the points on the curve $y = x^3 - 3x$ at which the tangent is parallel to the x-axis

$dy/dx = 3x^2 - 3 = 0 \rightarrow x^2 = 1 \rightarrow x = \pm 1$

At $x = 1$: $y = 1^3 - 3(1) = -2 \rightarrow (1, -2)$

At $x = -1$: $y = (-1)^3 - 3(-1) = -1 + 3 = 2 \rightarrow (-1, 2)$

Answer: Points: (1, -2), (-1, 2)

6. (a) In the diagram below, O is the origin, ABC is a straight line and M is the mid-point of OA.

IF OA = a, OB = b and AC = 3AB, find in terms of a and or b, in their simplest form:

(i) MA

M is the midpoint of OA, so the position vector of M is:

OM = OA / 2 = a / 2

The position vector of A is OA = a.

Therefore, the vector MA is:

MA = OA - OM = a - a/2 = a/2

Answer for (i): MA = a/2

(ii) AC

We are given AC = 3AB.

First, compute AB:

AB = OB - OA = b - a

Therefore:

AC = 3AB = 3(b - a) = 3b - 3a

Answer for (ii): AC = 3b - 3a

(iii) the position vector of C.

The position vector of A is OA = a.

The vector AC takes us from A to C, so the position vector of C (i.e., OC) is:

OC = OA + AC

From part (ii), AC = 3b - 3a. So:

OC = a + (3b - 3a) = a + 3b - 3a = 3b - 2a

Answer for (iii): Position vector of C = 3b - 2a

(b) A, B and C are the points with position vectors 2i - j + 5k, i - 2j + k and 3i + j - 2k respectively. IF D and E are the respective midpoints of BC and AC, show that DE is parallel to AB.

First, compute AB:

AB = OB - OA = (i - 2j + k) - (2i - j + 5k) = (1 - 2)i + (-2 - (-1))j + (1 - 5)k = -i - j - 4k

Find the position vector of D, the midpoint of BC:

OD = (OB + OC) / 2 = ((i - 2j + k) + (3i + j - 2k)) / 2 = ((1 + 3)i + (-2 + 1)j + (1 - 2)k) / 2 = (4i - j - k) / 2 = 2i - (1/2)j - (1/2)k

Find the position vector of E, the midpoint of AC:

OE = (OA + OC) / 2 = ((2i - j + 5k) + (3i + j - 2k)) / 2 = ((2 + 3)i + (-1 + 1)j + (5 - 2)k) / 2 = (5i + 0j + 3k) / 2 = (5/2)i + (3/2)k

Compute DE:

DE = OE - OD = ((5/2)i + (3/2)k) - (2i - (1/2)j - (1/2)k) = (5/2 - 2)i + (0 - (-1/2))j + (3/2 - (-1/2))k = (1/2)i + (1/2)j + 2k

Now compare DE with AB:

AB = -i - j - 4k

DE = (1/2)i + (1/2)j + 2k

Check if DE is a scalar multiple of AB:

Let DE = k x AB:

(1/2)i + (1/2)j + 2k = k(-i - j - 4k)

Equate coefficients:

For i: 1/2 = -k => k = -1/2

For j: 1/2 = -k => k = -1/2

For k: 2 = -4k => k = -1/2

Since k = -1/2 is consistent, DE = (-1/2)AB, so DE is parallel to AB.

Answer for (b): DE is parallel to AB.

(c) IF 4 [1 2] + 2 [1 1] = 3 [n -6], find the values of m and n.

[3 m] [-6 m] [-6 m]

Compute the left-hand side:

4 [1 2] + 2 [1 1] = [41 42] + [21 21] = [4 8] + [2 2] = [4+2 8+2] = [6 10]

[3 m] [-6 m] [43 4m] [2x-6 2m] [12 4m] [-12 2m] [12-12 4m+2m] [0 6m]

So the equation becomes:

[6 10] = 3 [n -6]

[0 6m] [-6 m]

Equate the elements:

First row, first column: 6 = 3n => n = 2

First row, second column: 10 = 3(-6) => 10 = -18 (not possible, but let's check others)

Second row, first column: 0 = 3(-6) => 0 = -18 (not possible)

Second row, second column: 6m = 3m => 6m - 3m = 0 => 3m = 0 => m = 0

The equation 10 = -18 and 0 = -18 indicates inconsistency, but based on the solvable parts:

m = 0, n = 2

Answer for (c): m = 0, n = 2