THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

141 BASIC APPLIED MATHEMATICS

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2011**

Instructions

1. This paper consists of **Ten (10)** questions.
2. Answer all questions.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

Prepared by: Maria Marco for TETEA

1. (a) Show that the distance between (4, 1) and (10,9) is equivalent to 10 units

Distance formula: $d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$

Points: (4, 1) and (10, 9)

$d = \sqrt{[(10 - 4)^2 + (9 - 1)^2]} = \sqrt{[6^2 + 8^2]} = \sqrt{(36 + 64)} = \sqrt{100}$

Answer: Distance between (4, 1) and (10, 9) = $\sqrt{[(10 - 4)^2 + (9 - 1)^2]} = \sqrt{100} = 10$ units

(b) Find the equation of a line, in the form of $ax + by + c = 0$, through the point (1, -2) which is perpendicular to $2y = 4x + 8$

Line: $2y = 4x + 8 \rightarrow y = 2x + 4 \rightarrow$ slope $m_1 = 2$

Perpendicular slope: $m_2 = -1/m_1 = -1/2$

Line through (1, -2) with slope -1/2: $y - (-2) = (-1/2)(x - 1)$

$y + 2 = (-1/2)x + 1/2$

Multiply by 2: $2y + 4 = -x + 1$

$x + 2y + 3 = 0$

Answer: $x + 2y + 3 = 0$

2. (a) A quadratic equation has positive roots α and β such that $\alpha - \beta = 2$ and $\alpha\beta = 15$. Determine its equation, and hence obtain the quadratic equation, whose roots are α^2 and β^2

Given: $\alpha + \beta$ (sum of roots), $\alpha\beta = 15$ (product), $\alpha - \beta = 2$

Let $\alpha + \beta = s$: $(\alpha + \beta)^2 = (\alpha - \beta)^2 + 4\alpha\beta \rightarrow s^2 = 2^2 + 4(15) = 4 + 60 = 64 \rightarrow s = 8$ (since roots are positive)

Equation: $x^2 - (\alpha + \beta)x + \alpha\beta = 0 \rightarrow x^2 - 8x + 15 = 0$

Roots of new equation: α^2 and β^2

$\alpha + \beta = 8, \alpha - \beta = 2 \rightarrow 2\alpha = 10 \rightarrow \alpha = 5, \beta = 3$

$\alpha^2 = 25, \beta^2 = 9$

Sum: $\alpha^2 + \beta^2 = 25 + 9 = 34$

Product: $\alpha^2\beta^2 = 25 \times 9 = 225$

New equation: $x^2 - 34x + 225 = 0$

Answer: Original equation: $x^2 - 8x + 15 = 0$; New equation: $x^2 - 34x + 225 = 0$

(b) Given the functions $f(x) = 2x - 5$ and $g(x) = 4x + 7$, verify that $[f \circ g]^{-1}(x) = g^{-1} \circ f^{-1}(x)$

$f \circ g(x) = f(g(x)) = f(4x + 7) = 2(4x + 7) - 5 = 8x + 14 - 5 = 8x + 9$

Inverse of $f \circ g$: $y = 8x + 9 \rightarrow x = (y - 9)/8 \rightarrow [f \circ g]^{-1}(x) = (x - 9)/8$

$f^{-1}(x)$: $y = 2x - 5 \rightarrow x = (y + 5)/2 \rightarrow f^{-1}(x) = (x + 5)/2$

$g^{-1}(x)$: $y = 4x + 7 \rightarrow x = (y - 7)/4 \rightarrow g^{-1}(x) = (x - 7)/4$

$g^{-1} \circ f^{-1}(x) = g^{-1}(f^{-1}(x)) = g^{-1}((x + 5)/2) = [((x + 5)/2) - 7]/4 = [(x + 5 - 14)/2]/4 = [(x - 9)/2]/4 = (x - 9)/8$

$[f \circ g]^{-1}(x) = g^{-1} \circ f^{-1}(x)$, verified.

Answer: $[f \circ g]^{-1}(x) = (x - 9)/8 = g^{-1} \circ f^{-1}(x)$, verified

3.(a) Solve the simultaneous equations $3x - y = -2$ and $x^2 + xy + y = 28$

From $3x - y = -2$: $y = 3x + 2$

Substitute into second: $x^2 + x(3x + 2) + (3x + 2) = 28$

$x^2 + 3x^2 + 2x + 3x + 2 = 28$

$4x^2 + 5x + 2 - 28 = 0$

$4x^2 + 5x - 26 = 0$

Solve: $x = [-5 \pm \sqrt{(5^2 - 4(4)(-26))}]/(2\times4) = [-5 \pm \sqrt{(25 + 416)}]/8 = (-5 \pm \sqrt{441})/8 = (-5 \pm 21)/8$

$x = 16/8 = 2$ or $x = -26/8 = -13/4$

$x = 2 \rightarrow y = 3(2) + 2 = 8 \rightarrow (2, 8)$

$x = -13/4 \rightarrow y = 3(-13/4) + 2 = -39/4 + 8/4 = -31/4 \rightarrow (-13/4, -31/4)$

Answer: $(x, y) = (2, 8)$ or $(-13/4, -31/4)$

(b) The first term of an Arithmetic Progression (A.P) is -12, and the last term is 40. If the sum of the progression is 196, find the number of terms and the common difference

First term $a = -12$, last term $l = 40$, sum $S_n = 196$

Sum: $S_n = (n/2)(a + l) = (n/2)(-12 + 40) = (n/2)(28) = 14n$

$14n = 196 \rightarrow n = 14$

Last term: $l = a + (n-1)d \rightarrow 40 = -12 + (14-1)d \rightarrow 40 = -12 + 13d$

$52 = 13d \rightarrow d = 4$

Answer: Number of terms = 14, common difference = 4

4. (a) The length (l) of a simple pendulum varies as the square of its period (T). The time to swing to and fro. A pendulum 0.994 m long has a period of approximately 2 seconds. Find:

(i) the length of a pendulum whose period is 3 seconds

$l \propto T^2 \rightarrow l = k T^2$

At $l = 0.994$ m, $T = 2$ s: $0.994 = k (2^2) \rightarrow 0.994 = 4k \rightarrow k = 0.994/4 = 0.2485$

For $T = 3$ s: $l = 0.2485 (3^2) = 0.2485 \times 9 = 2.2365$ m

Answer: (i) Length = 2.2365 m

(ii) an equation connecting l and T

From above: $l = 0.2485 T^2$

Answer: (ii) $l = 0.2485 T^2$

(b) A traveler in Uganda changed Tshs 2,000,000/- into Uganda shillings (Ushs) at a rate of Tshs 1= Ushs 2. He spent Ushs 2,500,000/- and then he changed the rest back into Tshs at the rate of Tshs 1= Ushs 2.5. How much Tanzanian shillings did he receive?

Tshs 2,000,000 at Tshs 1 = Ushs 2 → Ushs 2,000,000 × 2 = Ushs 4,000,000

Spent Ushs 2,500,000 → Remaining: Ushs 4,000,000 - 2,500,000 = Ushs 1,500,000

Convert back at Tshs 1 = Ushs 2.5: Ushs 1,500,000 / 2.5 = Tshs 600,000

Answer: Tshs 600,000

5. (a) Prove that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$

Left: $\sin(A + B)\sin(A - B) = (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$

$= (\sin A \cos B)^2 - (\cos A \sin B)^2 = \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$

Right: $\sin^2 A - \sin^2 B = \sin^2 A - \sin^2 B$

Use identity: $\sin^2 A = 1 - \cos^2 A \rightarrow \sin^2 A - \sin^2 B = (1 - \cos^2 A) - (1 - \cos^2 B) = \cos^2 B - \cos^2 A$

Left: $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = (1 - \cos^2 A) \cos^2 B - \cos^2 A (1 - \cos^2 B) = \cos^2 B - \cos^2 A \cos^2 B - \cos^2 A + \cos^2 A \cos^2 B = \cos^2 B - \cos^2 A$

Left = Right, proven.

Answer: $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$ (proven)

(b) In the triangle below calculate the size of angle Y

Triangle XYZ: XY = 4.5, YZ = 3.5, XZ = 6.5

Use Law of Cosines at angle Y: $\cos Y = (XY^2 + YZ^2 - XZ^2)/(2 \times XY \times YZ)$

$= (4.5^2 + 3.5^2 - 6.5^2)/(2 \times 4.5 \times 3.5) = (20.25 + 12.25 - 42.25)/(31.5) = -9.75/31.5 \approx -0.3095$

$Y = \cos^{-1}(-0.3095) \approx 108°$

Answer: Angle Y ≈ 108°

6. (a) Solve each of the following equations:

(i) $\log x + \log 2 - \log 7 = 1$

$\log x + \log 2 - \log 7 = \log (x \times 2 / 7) = 1$

$x \times 2 / 7 = 10^1 = 10$

$x \times 2 = 70 \rightarrow x = 35$

Answer: (i) x = 35

(ii) $\log (x + 1) - \log (x - 2) = 2$

$\log [(x + 1)/(x - 2)] = 2$

$(x + 1)/(x - 2) = 10^2 = 100$

$x + 1 = 100(x - 2)$

$x + 1 = 100x - 200$

$201 = 99x \rightarrow x = 201/99 = 67/33$

Answer: (ii) x = 67/33

(b) Using scientific notation, evaluate 34000 × 0.00538 / 0.027 × 430000 rounding up to three decimal places

$34000 \times 0.00538 / (0.027 \times 430000)$

$= (3.4 \times 10^4) \times (5.38 \times 10^{-3}) / [(2.7 \times 10^{-2}) \times (4.3 \times 10^5)]$

Numerator: $3.4 \times 5.38 = 18.292 \times 10^{4-3} = 18.292 \times 10^1$

Denominator: $2.7 \times 4.3 = 11.61 \times 10^{-2+5} = 11.61 \times 10^3$

$18.292 \times 10^1 / (11.61 \times 10^3) = (18.292 / 11.61) \times 10^{1-3} = 1.575 \times 10^{-2} = 0.01575$

Round to 3 decimal places: 0.016

Answer: 0.016

7. (a) Differentiate $(x - 6)/(x + 5)^2$

Use the quotient rule: $(u/v)' = (u'v - uv')/v^2$

$u = x - 6$, $v = (x + 5)^2$

$u' = 1$, $v' = 2(x + 5)$

$dy/dx = [(1)(x + 5)^2 - (x - 6)(2(x + 5))] / [(x + 5)^2]^2$

$= [(x + 5)^2 - 2(x - 6)(x + 5)] / (x + 5)^4$

Factor: $(x + 5) [(x + 5) - 2(x - 6)] / (x + 5)^4$

$= (x + 5 - 2x + 12) / (x + 5)^3 = (17 - x) / (x + 5)^3$

Answer: $dy/dx = (17 - x) / (x + 5)^3$

(b) A container in the shape of a right circular cone of height 20 cm and base radius 2 cm is catching the drips from a tap leaking at the rate of $0.3\ cm^3\ s^{-1}$. Find the rate at which the surface area of water is increasing when the water is half way up the cone

Cone: height h = 20 cm, base radius r = 2 cm

Volume of cone: $V = (1/3)\pi r^2 h$

At half height (h = 10 cm), find the radius of the water surface:

Ratio of radii to heights: $r/h = 2/20 = 0.1 \rightarrow$ at h = 10 cm, $r = 0.1 \times 10 = 1$ cm

Surface area of water (circular): $A = \pi r^2 = \pi (0.1h)^2 = \pi (0.01h^2) = 0.01\pi h^2$

$dA/dh = 0.01\pi \times 2h = 0.02\pi h$

Volume of water: $V = (1/3)\pi r^2 h = (1/3)\pi (0.1h)^2 h = (1/3)\pi (0.01h^2) h = (0.01/3)\pi h^3$

$dV/dh = (0.01/3)\pi \times 3h^2 = 0.01\pi h^2$

Given $dV/dt = 0.3$ cm³ s⁻¹, use chain rule: $dV/dt = (dV/dh)(dh/dt)$

$0.3 = 0.01\pi h^2 (dh/dt)$

At h = 10 cm: $0.3 = 0.01\pi (10)^2 (dh/dt) = \pi (dh/dt)$

$dh/dt = 0.3/\pi$ cm s^{-1}

$dA/dt = (dA/dh)(dh/dt) = (0.02\pi h)(0.3/\pi) = 0.02 \times 0.3 \times h = 0.006 h$

At h = 10 cm: $dA/dt = 0.006 \times 10 = 0.06$ cm^2 s^{-1}

Answer: Rate of increase of surface area = 0.06 cm^2 s^{-1}

8. (a) Find $\int \cos x \sin x \, dx$

Use substitution: Let $u = \sin x \rightarrow du = \cos x \, dx$

$\int \cos x \sin x \, dx = \int u \, du = u^2/2 + C = (\sin^2 x)/2 + C$

Alternatively, use identity: $\cos x \sin x = (1/2) \sin 2x$

$\int (1/2) \sin 2x \, dx = (1/2) \times (-\cos 2x)/2 + C = -(\cos 2x)/4 + C$

Both forms are equivalent (verify using $\sin^2 x = (1 - \cos 2x)/2$).

Answer: $\int \cos x \sin x \, dx = (\sin^2 x)/2 + C$

(b) Evaluate $\int$ (from 1 to 3) $[x^2/\sqrt{(x^2 + 3)}] \, dx$, leaving your answer in surd form

Substitute $u = x^2 + 3 \rightarrow du = 2x \, dx \rightarrow x \, dx = du/2$

When x = 1, u = 4; when x = 3, u = 12

$\int x^2/\sqrt{(x^2 + 3)} \, dx = \int (u - 3)/\sqrt{u} \, (du/2) = (1/2) \int (u - 3)/u^{(1/2)} \, du = (1/2) \int (u^{(1/2)} - 3u^{(-1/2)}) \, du$

$= (1/2) [(2/3)u^{(3/2)} - 3(2)u^{(1/2)}] = (1/3)u^{(3/2)} - 3u^{(1/2)} + C$

From 1 to 3: $[(1/3)(12)^{(3/2)} - 3(12)^{(1/2)}] - [(1/3)(4)^{(3/2)} - 3(4)^{(1/2)}]$

$12^{(3/2)} = (12)^{(1/2)} \times 12 = 2\sqrt{3} \times 12 = 24\sqrt{3}$

$12^{(1/2)} = 2\sqrt{3}$

$4^{(3/2)} = (4)^{(1/2)} \times 4 = 2 \times 4 = 8$

$4^{(1/2)} = 2$

$= [(1/3)(24\sqrt{3}) - 3(2\sqrt{3})] - [(1/3)(8) - 3(2)]$

$= [8\sqrt{3} - 6\sqrt{3}] - [8/3 - 6] = 2\sqrt{3} - (8/3 - 18/3) = 2\sqrt{3} - (-10/3) = 2\sqrt{3} + 10/3$

Answer: $2\sqrt{3} + 10/3$

9. (a) Given that $a = 4i + 3j + 12k$ and $b = 8i - 6j$, find a^2, b^2 and hence determine the angle between the vectors a and b

$a^2 = |a|^2 = 4^2 + 3^2 + 12^2 = 16 + 9 + 144 = 169$

$b^2 = |b|^2 = 8^2 + (-6)^2 + 0^2 = 64 + 36 = 100$

Angle θ: $\cos \theta = (a \cdot b) / (|a| |b|)$

$a \cdot b = (4)(8) + (3)(-6) + (12)(0) = 32 - 18 = 14$

$|a| = \sqrt{169} = 13$, $|b| = \sqrt{100} = 10$

$\cos \theta = 14 / (13 \times 10) = 14/130 = 7/65$

$\theta = \cos^{-1}(7/65) \approx 84.2°$

Answer: $a^2 = 169$, $b^2 = 100$, angle $\theta \approx 84.2°$

(b) If A and B are points (1, 1, 1) and (13, 4, 5) respectively, find the displacement vector AB and hence the unit vector parallel to AB

AB = B - A = (13 - 1, 4 - 1, 5 - 1) = (12, 3, 4)

Magnitude: $|AB| = \sqrt{(12^2 + 3^2 + 4^2)} = \sqrt{(144 + 9 + 16)} = \sqrt{169} = 13$

Unit vector: AB / |AB| = (12/13, 3/13, 4/13)

Answer: AB = (12, 3, 4), unit vector = (12/13, 3/13, 4/13)

10. (a) Calculate the standard deviation of the numbers 9, 3, 8, 8, 9, 8, 9, 18

Mean: (9 + 3 + 8 + 8 + 9 + 8 + 9 + 18) / 8 = 72 / 8 = 9

Variance: $\Sigma(x - \text{mean})^2 / n$

$(9 - 9)^2 = 0$ (4 times), $(3 - 9)^2 = 36$, $(8 - 9)^2 = 1$ (3 times), $(18 - 9)^2 = 81$

Sum = (0×4) + 36 + (1×3) + 81 = 36 + 3 + 81 = 120

Variance = 120 / 8 = 15

Standard deviation = $\sqrt{15} \approx 3.873$

Answer: Standard deviation ≈ 3.873

(b) Find the range of the numbers 51.6, 48.7, 50.3, 49.5, and 48.9

Max = 51.6, Min = 48.7

Range = 51.6 - 48.7 = 2.9

Answer: Range = 2.9

(c) Calculate the mean of the distribution of marks given below:

Marks | Frequency

0 - 9 | 0

10 - 19 | 3

20 - 29 | 7

30 - 39 | 12

40 - 49 | 18

50 - 59 | 22

60 - 69 | 17

70 - 79 | 14

80 - 89 | 9

90 - 99 | 5

Midpoints: 4.5, 14.5, 24.5, 34.5, 44.5, 54.5, 64.5, 74.5, 84.5, 94.5

Total frequency = 0 + 3 + 7 + 12 + 18 + 22 + 17 + 14 + 9 + 5 = 107

Sum = (14.5×3) + (24.5×7) + (34.5×12) + (44.5×18) + (54.5×22) + (64.5×17) + (74.5×14) + (84.5×9) + (94.5×5)

= 43.5 + 171.5 + 414 + 801 + 1199 + 1096.5 + 1043 + 760.5 + 472.5 = 6001

Mean = 6001 / 107 ≈ 56.084

11. (a) A fair die is thrown once. List the possible outcomes and hence evaluate the probability of scoring a multiple of 2

Possible outcomes: {1, 2, 3, 4, 5, 6}

Multiples of 2: {2, 4, 6} → 3 outcomes

Total outcomes: 6

Probability = 3/6 = 1/2

Answer: Outcomes: {1, 2, 3, 4, 5, 6}; Probability = 1/2

(b) The events A and B are such that P(A) = 0.43, P(B) = 0.48 and P(A ∪ B) = 0.78. Show that the events A and B are not independent

For independence: P(A ∩ B) = P(A)P(B)

P(A)P(B) = 0.43 × 0.48 = 0.2064

Use: P(A ∪ B) = P(A) + P(B) - P(A ∩ B)

0.78 = 0.43 + 0.48 - P(A ∩ B)

P(A ∩ B) = 0.91 - 0.78 = 0.13

Since P(A ∩ B) = 0.13 ≠ 0.2064, A and B are not independent.

Answer: P(A ∩ B) = 0.13 ≠ P(A)P(B) = 0.2064, so A and B are not independent

(c) In how many different ways can eight cards be dealt from a pack of fifty-two playing cards?

Number of ways to choose 8 cards from 52: C(52, 8)

C(52, 8) = 52! / (8! × 44!) = (52 × 51 × 50 × 49 × 48 × 47 × 46 × 45) / (8 × 7 × 6 × 5 × 4 × 3 × 2 × 1)

= (52 × 51 × 50 × 49 × 48 × 47 × 46 × 45) / 40320

Compute numerator: 52 × 51 = 2652, × 50 = 132600, × 49 = 6497400, × 48 = 311875200, × 47 = 14658134400, × 46 = 674274182400, × 45 = 30342338208000

C(52, 8) = 30342338208000 / 40320 = 752539230

Answer: 752539230 ways

12. (a) Find the product AB when

A = [1 0 1] and B = [4 -1 3]

[2 1 3]   [2 2 2]

[4 2 1]   [3 7 1]

A is 3×3, B is 3×3 → AB is 3×3

AB = [(1×4 + 0×2 + 1×3) (1×-1 + 0×2 + 1×7) (1×3 + 0×2 + 1×1)]

[(2×4 + 1×2 + 3×3) (2×-1 + 1×2 + 3×7) (2×3 + 1×2 + 3×1)]

[(4×4 + 2×2 + 1×3) (4×-1 + 2×2 + 1×7) (4×3 + 2×2 + 1×1)]

= [7 6 4]

[19 21 11]

[23 7 17]

Answer: AB = [7 6 4]

[19 21 11]

[23 7 17]

(b) If A = [2 1 1] and B = [1 -1 0], find a matrix X such that AX + B = A

[1 0 1]   [0 0 -1]

[0 -1 0]   [1 2 -1]

AX + B = A

AX = A - B

A = [2 1 1], B = [1 -1 0] → A - B = [1 2 1]

[1 0 1]   [0 0 -1]   [1 0 2]

[0 -1 0]   [1 2 -1]   [-1 -3 1]

Solve AX = A - B for X: X = A^{-1} (A - B)

Det(A) = 2(0×0 - 1×-1) - 1(1×0 - 1×0) + 1(1×-1 - 0×0) = 2(1) - 1(0) + 1(-1) = 1

Adjoint of A:

Cofactors: C_{11} = 1, C_{12} = 0, C_{13} = -1, C_{21} = -1, C_{22} = 2, C_{23} = 1, C_{31} = 1, C_{32} = -2, C_{33} = -1

Adj(A) = [1 -1 1]

[0 2 -2]

[-1 1 -1]

A^{-1} = Adj(A)/Det(A) = [1 -1 1]

[0 2 -2]

[-1 1 -1]

X = A^{-1} (A - B) = [1 -1 1] [1 2 1] = [0 1 2]

[0 2 -2] [1 0 2] [2 4 -4]

[-1 1 -1] [-1 -3 1] [2 -2 -2]

Answer: X = [0 1 2]

[2 4 -4]

[2 -2 -2]

(c) Solve the equations 2x + 3y = 8 and 5x - 2y = 1 by using the inverse matrix method

Matrix form: AX = B

A = [2 3], X = [x], B = [8]

[5 -2] [y] [1]

Det(A) = (2)(-2) - (3)(5) = -4 - 15 = -19

Adjoint of A: C_{11} = -2, C_{12} = -5, C_{21} = -3, C_{22} = 2 → Adj(A) = [-2 -3]

[-5 2]

A^{-1} = (1/-19) [-2 -3] = [2/19 3/19]

[-5 2] [5/19 -2/19]

X = A^{-1} B = [2/19 3/19] [8] = [(2/19)(8) + (3/19)(1)] = [19/19] = [1]

[5/19 -2/19] [1] [(5/19)(8) + (-2/19)(1)] [38/19] [2]

Answer: x = 1, y = 2

13.

Solve the linear programming problem:

Maximize x - (3/2)y subject to the constraints:

$2x + 4y \leq 12$

$3x + 2y \leq 10$

$x, y \geq 0$

Objective: Maximize $P = x - (3/2)y$

Constraints:

$2x + 4y \leq 12 \rightarrow x + 2y \leq 6$

$3x + 2y \leq 10$

$x, y \geq 0$

Graph:

$x + 2y \leq 6 \rightarrow (6, 0), (0, 3)$

$3x + 2y \leq 10 \rightarrow (10/3, 0), (0, 5)$

Vertices:

(0, 0)

(0, 3) (from $x + 2y = 6$, $x = 0$)

(10/3, 0) (from $3x + 2y = 10$, $y = 0$)

Intersection: $x + 2y = 6$, $3x + 2y = 10 \rightarrow 2x = 4 \rightarrow x = 2$, $2 + 2y = 6 \rightarrow y = 2 \rightarrow (2, 2)$

Evaluate P:

(0, 0): $P = 0 - (3/2)(0) = 0$

(0, 3): $P = 0 - (3/2)(3) = -4.5$

(10/3, 0): $P = 10/3 - (3/2)(0) = 10/3 \approx 3.333$

(2, 2): $P = 2 - (3/2)(2) = 2 - 3 = -1$

Maximum at (10/3, 0).

Answer: Maximum $P = 10/3$ at $(x, y) = (10/3, 0)$

14.

(a) Differentiate $f(x) = 1/\sqrt{x}$ from first principle

$f(x) = 1/\sqrt{x} = x^{(-1/2)}$

First principle: $f'(x) = \lim_{h\to 0} [f(x + h) - f(x)]/h$

$= \lim_{h\to 0} [(x + h)^{(-1/2)} - x^{(-1/2)}]/h$

Multiply by conjugate: $[(x + h)^{-1/2} - x^{-1/2}][(x + h)^{-1/2} + x^{-1/2}] / [h [(x + h)^{-1/2} + x^{-1/2}]]$

Numerator: $(x + h)^{-1} - x^{-1} = (x - (x + h))/(x(x + h)) = -h/(x(x + h))$

Denominator: $h [(x + h)^{-1/2} + x^{-1/2}]$

$= [-h/(x(x + h))] / [h ((x + h)^{-1/2} + x^{-1/2})] = -1 / [x(x + h) ((x + h)^{-1/2} + x^{-1/2})]$

As $h \to 0$: $-1 / [x(x)(x^{-1/2} + x^{-1/2})] = -1 / (x^2 (2x^{-1/2})) = -1 / (2x^{3/2})$

Answer: $f'(x) = -1 / (2x^{3/2})$

(b) Determine dy/dx given that $y^3 + x^3 + \cos(x + y) = 0$

Implicit differentiation:

$3y^2\, dy/dx + 3x^2 - \sin(x + y)(1 + dy/dx) = 0$

$3y^2\, dy/dx - \sin(x + y) - \sin(x + y)\, dy/dx + 3x^2 = 0$

$(3y^2 - \sin(x + y))\, dy/dx = \sin(x + y) - 3x^2$

$dy/dx = [\sin(x + y) - 3x^2] / [3y^2 - \sin(x + y)]$

Answer: $dy/dx = [\sin(x + y) - 3x^2] / [3y^2 - \sin(x + y)]$

(c) Solve for the stationary values of the function $x^3 - 2x + 11 = 0$

Function: $f(x) = x^3 - 2x + 11$

Stationary points: $f'(x) = 0$

$f'(x) = 3x^2 - 2 = 0 \to x^2 = 2/3 \to x = \pm\sqrt{(2/3)} = \pm\sqrt{6}/3$

Second derivative: $f''(x) = 6x$

At $x = \sqrt{6}/3$: $f'' = 6(\sqrt{6}/3) > 0 \to$ minimum

At $x = -\sqrt{6}/3$: $f'' = 6(-\sqrt{6}/3) < 0 \to$ maximum

Values:

$x = \sqrt{6}/3$: $f = (\sqrt{6}/3)^3 - 2(\sqrt{6}/3) + 11 = (\sqrt{6^3}/27) - 2\sqrt{6}/3 + 11 = 6\sqrt{6}/27 - 2\sqrt{6}/3 + 11 = (2\sqrt{6} - 6\sqrt{6})/9 + 11 = (-4\sqrt{6})/9 + 11$

$x = -\sqrt{6}/3$: $f = (-\sqrt{6}/3)^3 - 2(-\sqrt{6}/3) + 11 = -2\sqrt{6}/9 + 2\sqrt{6}/3 + 11 = (-2\sqrt{6} + 6\sqrt{6})/9 + 11 = (4\sqrt{6})/9 + 11$

Answer: Stationary values: minimum $(-4\sqrt{6})/9 + 11$ at $x = \sqrt{6}/3$, maximum $(4\sqrt{6})/9 + 11$ at $x = -\sqrt{6}/3$

15. (a) Calculate the area enclosed between the curve $y = x(x - 1)(x - 2)$ and the x-axis

$y = x(x - 1)(x - 2) = x^3 - 3x^2 + 2x$

x-intercepts: $x = 0$, $x = 1$, $x = 2$

Between $x = 0$ and 1, $y \geq 0$; between $x = 1$ and 2, $y \leq 0$.

Area = ∫ (from 0 to 1) $(x^3 - 3x^2 + 2x)$ dx + ∫ (from 1 to 2) $-(x^3 - 3x^2 + 2x)$ dx

$\int (x^3 - 3x^2 + 2x)\, dx = (x^4/4) - x^3 + x^2$

From 0 to 1: $[(1/4) - 1 + 1] - [0] = 1/4$

From 1 to 2: $-[(16/4) - 8 + 4] + [(1/4) - 1 + 1] = -[0] + [1/4] = 1/4$

Total area = $1/4 + 1/4 = 1/2$

Answer: Area = 1/2

(b) Evaluate the integral ∫ (from 3 to 5) $(x^3/\sqrt{(x^2 + 3)})$ dx

Substitute $u = x^2 + 3 \rightarrow du = 2x\, dx \rightarrow x\, dx = du/2$

$x^2 = u - 3 \rightarrow x^3 = x^2 \times x = (u - 3)\sqrt{u}$

When $x = 3$, $u = 12$; when $x = 5$, $u = 28$

$\int x^3/\sqrt{(x^2 + 3)}\, dx = \int (u - 3)/\sqrt{u}\, (du/2) = (1/2) \int (u - 3)/u^{(1/2)}\, du = (1/2) \int (u^{(1/2)} - 3u^{(-1/2)})\, du$

$= (1/2) [(2/3)u^{(3/2)} - 3(2)u^{(1/2)}] = (1/3)u^{(3/2)} - 3u^{(1/2)}$

From 12 to 28: $[(1/3)(28)^{(3/2)} - 3(28)^{(1/2)}] - [(1/3)(12)^{(3/2)} - 3(12)^{(1/2)}]$

$28^{(3/2)} = (28)^{(1/2)} \times 28 = 2\sqrt{7} \times 28 = 56\sqrt{7}$

$28^{(1/2)} = 2\sqrt{7}$

$12^{(3/2)} = (12)^{(1/2)} \times 12 = 2\sqrt{3} \times 12 = 24\sqrt{3}$

$12^{(1/2)} = 2\sqrt{3}$

$= [(1/3)(56\sqrt{7}) - 3(2\sqrt{7})] - [(1/3)(24\sqrt{3}) - 3(2\sqrt{3})]$

$= [(56\sqrt{7})/3 - 6\sqrt{7}] - [(24\sqrt{3})/3 - 6\sqrt{3}] = (14\sqrt{7})/3 - (2\sqrt{3})/3$

Answer: $(14\sqrt{7} - 2\sqrt{3})/3$

(c) What is the volume generated when the area enclosed by the curve $y = x$, the x-axis and the line $x = 2$ is rotated about the x-axis?

$y = x$, from $x = 0$ to $x = 2$

Volume $= \pi \int$ (from 0 to 2) $y^2\,dx = \pi \int$ (from 0 to 2) $x^2\,dx$

$= \pi\,[x^3/3]$ (from 0 to 2) $= \pi\,[(8/3) - 0] = (8\pi)/3$

Answer: Volume $= (8\pi)/3$

16. (a) Write down the unit vector which is perpendicular to the plane $4x + 3y + 2z = 12$

Plane: $4x + 3y + 2z = 12$

Normal vector: (4, 3, 2)

Magnitude: $\sqrt{(4^2 + 3^2 + 2^2)} = \sqrt{(16 + 9 + 4)} = \sqrt{29}$

Unit vector: $(4/\sqrt{29}, 3/\sqrt{29}, 2/\sqrt{29})$

Answer: Unit vector $= (4/\sqrt{29}, 3/\sqrt{29}, 2/\sqrt{29})$

(b) Find the equation of a plane through the point (2, 4, 5) and perpendicular to the vector $2i + 7j + 8k$

Normal vector: (2, 7, 8)

Plane equation: $2(x - 2) + 7(y - 4) + 8(z - 5) = 0$

$2x - 4 + 7y - 28 + 8z - 40 = 0$

$2x + 7y + 8z - 72 = 0$

Answer: $2x + 7y + 8z - 72 = 0$

(c) Compute the perpendicular distance of the point P(0, 14, 10) from the line whose equation is $r = (i + 2j + 3k) + \lambda(3i + 4k)$

Line: $r = (1, 2, 3) + \lambda(3, 0, 4)$

Direction vector: (3, 0, 4)

Point on line: (1, 2, 3)

Vector AP (from (1, 2, 3) to (0, 14, 10)): (-1, 12, 7)

Distance = |(AP × direction vector)| / |direction vector|

$AP \times (3, 0, 4) = (12\times4 - 7\times0, -(7\times3 - (-1)\times4), -1\times0 - 12\times3) = (48, -25, -36)$

Magnitude of cross product: $\sqrt{48^2 + (-25)^2 + (-36)^2} = \sqrt{2304 + 625 + 1296} = \sqrt{4225} = 65$

Magnitude of direction vector: $\sqrt{3^2 + 0^2 + 4^2} = \sqrt{25} = 5$

Distance = 65 / 5 = 13

Answer: Distance = 13