THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

141 BASIC APPLIED MATHEMATICS

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2020**

Instructions

1. This paper consists of TEN questions.
2. Answer all questions.

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1. Use a non-programmable scientific calculator to compute:

(a) the value of $(3 + 3\sqrt{(0.65)}) / (3 - 3\sqrt{(0.65)})$ correct to 4 significant figures.
19.06

(b) the mean and standard deviation of 33, 28, 26, 35 and 38 correct to 2 decimal places.
Mean = 32.00
Standard deviation = 4.20

(c) the value of ${}^6C_2 + {}^7P_4 / 11!$ correct to 4 decimal places.
0.0085

2. (a) The function f is defined as

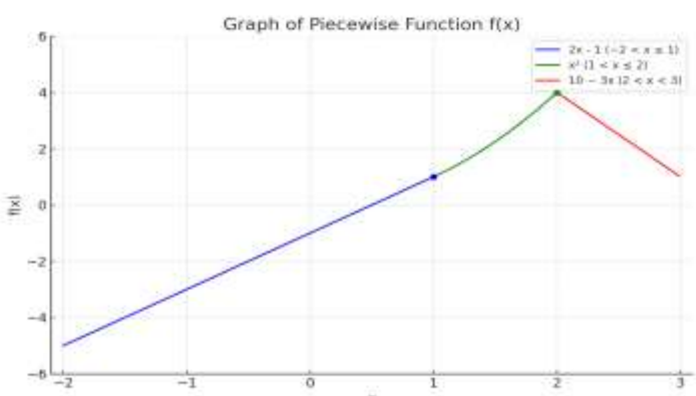
$$f(x) = \begin{cases} 2x - 1 & \text{if } -2 < x \leq 1 \\ x^2 & \text{if } 1 < x \leq 2 \\ 10 - 3x & \text{if } 2 < x < 3 \end{cases}$$

(i) Sketch the graph of f(x)

This is a piecewise function:

For $-2 < x \leq 1$: linear, slope = 2, intercept −1
Start at $x = -2$: $f(-2) = 2(-2) - 1 = -5$ (open circle)
End at $x = 1$: $f(1) = 2(1) - 1 = 1$ (closed point)
For $1 < x \leq 2$: quadratic x^2
At $x = 1$ (open): $f(1)$ not included
At $x = 2$: $f(2) = 4$ (closed point)

For $2 < x < 3$: $f(x) = 10 - 3x$ (decreasing linear)
At $x = 2$ (open): $f(2) = 4$
At $x = 3$ (open): $f(3) = 10 - 9 = 1$



(ii) State the domain and range of $f(x)$

Domain: $-2 < x < 3$

Range:

For $2x - 1 \rightarrow$ from $f(-2) \rightarrow -5$ up to $f(1) = 1$

For $x^2 \rightarrow$ from just above $x = 1$ to 4

For $10 - 3x \rightarrow$ from just below $x = 3 \rightarrow f(x)$ from 4 to 1

So overall range: $(-5, 4]$

(b) Given that $f(x) = 3x + 3$ and $g(x) = x + 3$, find:

(i) $(fog)(x)$

$fog(x) = f(g(x)) = f(x + 3) = 3(x + 3) + 3 = 3x + 9 + 3 = 3x + 12$

(ii) $(gof)(x)$

$gof(x) = g(f(x)) = g(3x + 3) = (3x + 3) + 3 = 3x + 6$

3. (a) Use the substitution method to solve the following system of equations:

$3x - y = 9$

$x^2 + xy + 2 = 0$

From first: $y = 3x - 9$

Substitute into second:

$x^2 + x(3x - 9) + 2 = 0$

$x^2 + 3x^2 - 9x + 2 = 0$

$4x^2 - 9x + 2 = 0$

Use quadratic formula:

$x = [9 \pm \sqrt{(81 - 32)}] / 8 = [9 \pm \sqrt{49}]/8$

$= [9 \pm 7]/8 \rightarrow x = 2, x = 0.25$

If $x = 2 \rightarrow y = 3\times2 - 9 = -3$

If $x = 0.25 \rightarrow y = 3\times0.25 - 9 = -8.25$

Solutions: $(2, -3)$ and $(0.25, -8.25)$

(b) Find the value(s) of x satisfying the equation $4^x - 6(2^x) - 16 = 0$

Let $2^x = y \rightarrow$ then $4^x = (2^2)^x = y^2$

Equation becomes: $y^2 - 6y - 16 = 0$

$y = [6 \pm \sqrt{(36 + 64)}] / 2 = [6 \pm \sqrt{100}]/2 = [6 \pm 10]/2$

$y = 8$ or -2

$2^x = 8 \rightarrow x = 3$

$2^x = -2 \rightarrow$ not valid (no real solution)
$x = 3$

(c) Find the sum of the first n terms of the series 1 + 3 + 5 + ...
This is an arithmetic sequence: a = 1, d = 2
$S_n = n/2 \times [2a + (n - 1)d] = n/2 \times [2 + 2n - 2] = n/2 \times 2n = n^2$
$S_n = n^2$

4. (a) Find the first derivative of $f(x) = x^2$ from first principles.

$f'(x) = \lim h \rightarrow 0\ [f(x + h) - f(x)] / h$
$= \lim h \rightarrow 0\ [(x + h)^2 - x^2]/h$
$= \lim h \rightarrow 0\ [x^2 + 2xh + h^2 - x^2]/h = \lim h \rightarrow 0\ [2xh + h^2]/h$
$= \lim h \rightarrow 0\ (2x + h) = 2x$

$f'(x) = 2x$

(b) Find the slope of the tangent to the curve $8x^2 + xy^2 - 5y^2 = 0$ at (1, −1)

Implicit differentiation:
$d/dx\ [8x^2 + xy^2 - 5y^2] = 0$
$16x + y^2 + x(2y\ dy/dx) - 10y\ dy/dx = 0$
$16x + y^2 + 2xy\ dy/dx - 10y\ dy/dx = 0$

Factor dy/dx:
$(2xy - 10y)\ dy/dx = -(16x + y^2)$
$dy/dx = -(16x + y^2) / (2xy - 10y)$

At (1, −1):
$dy/dx = -(16\times1 + (-1)^2) / (2\times1\times(-1) - 10\times(-1)) = -(16 + 1)/(-2 + 10) = -17/8$
Slope = −17/8

5. (a) Integrate $\sin^2 2x \cos 2x$ with respect to x.

Let $u = \sin 2x \rightarrow du/dx = 2\cos 2x$
So $\sin^2 2x \cos 2x\ dx = (1/2)\ u^2\ du$
$\int \sin^2 2x \cos 2x\ dx = (1/2) \int u^2\ du = (1/2) \times (u^3/3) = (1/6) \sin^3 2x + C$

Result: $(1/6) \sin^3 2x + C$

(b) Evaluate $\int_0^1 x / (x^2 + a)\ dx$ (express your answer in the form m ln n)

Let $u = x^2 + a \rightarrow du = 2x\ dx$

Then:

$\int x / (x^2 + a)\, dx = (1/2) \int du / u = (1/2) \ln|x^2 + a| + C$

$\int_0^1 x / (x^2 + a)\, dx = (1/2)[\ln(1 + a) - \ln(a)] = (1/2) \ln((1 + a)/a)$

So $m = 1/2$, $n = (1 + a)/a$

(c) Find the area enclosed between the curves $y = x^2 + 2$ and $y = 10 - x^2$

Area = $\int$ from a to b (upper − lower) dx

Set $x^2 + 2 = 10 - x^2 \rightarrow 2x^2 = 8 \rightarrow x^2 = 4 \rightarrow x = \pm 2$

Area = $\int_{-2}^{2} [(10 - x^2) - (x^2 + 2)]\, dx$

$= \int_{-2}^{2} [8 - 2x^2]\, dx$

$= 2\int_0^2 (8 - 2x^2)\, dx = 2[8x - (2/3)x^3]_0^2 = 2[(16 - (16/3))]$

$= 2[(48 - 16)/3] = 2(32/3) = 64/3$

Area = $64/3$ units^2

6. The following table shows litres of milk produced by 131 cows each day:

Litres of milk	5–10	11–16	17–22	23–28	29–34	35–40
No. of cows	15	28	37	26	18	7

(a) Estimate the mode.

Modal class = class with highest frequency = 17–22

$L = 16.5$, $f_1 = 37$, $f_0 = 28$, $f_2 = 26$, $h = 6$

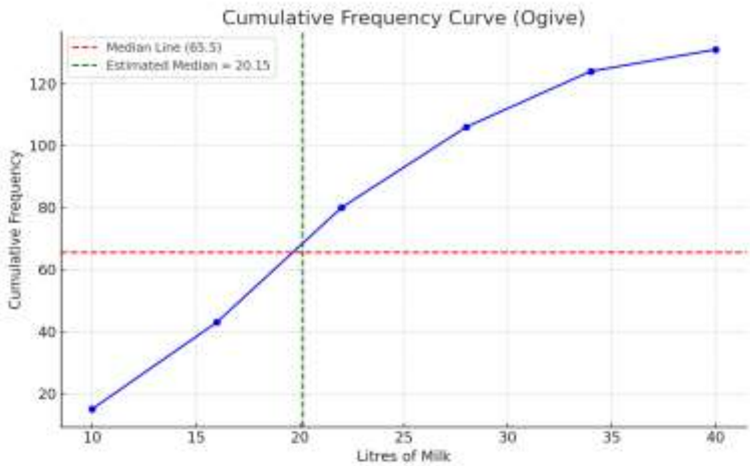
Mode = $L + [(f_1 - f_0) / (2f_1 - f_0 - f_2)] \times h$

$= 16.5 + [(37 - 28)/(2\times37 - 28 - 26)] \times 6$

$= 16.5 + [9 / (74 - 54)] \times 6 = 16.5 + (9/20)\times6 = 16.5 + 2.7 = 19.2$

Mode $\approx$ 19.2 litres

(b) Draw the cumulative frequency curve and use it to estimate the median.



Class boundaries:
Upper bounds: 10, 16, 22, 28, 34, 40
Cumulative frequencies:
15, 43, 80, 106, 124, 131

Median = 131/2 = 65.5th cow lies in class 17–22
L = 16.5, F = 43, f = 37, h = 6
Median = L + [(65.5 − 43)/37] × 6 = 16.5 + (22.5/37) × 6 ≈ 16.5 + 3.65 = 20.15

Median ≈ 20.15 litres

7. (a) One card is drawn at random from a well-shuffled pack of 52 playing cards. Find the probability that the card drawn is:

(i) a king or a queen
There are 4 kings and 4 queens → 4 + 4 = 8
P = 8/52 = 2/13

(ii) not a heart
There are 13 hearts out of 52 → not a heart = 39
P = 39/52 = 3/4

(iii) a red king
There are 2 red kings (hearts and diamonds)
P = 2/52 = 1/26

(b) A bag contains 3 red balls, 2 green balls and 5 blue balls. Two balls are drawn one after another without replacement. What is the probability that:

(i) both balls are red

First draw red: 3/10
Second draw red: 2/9
P = 3/10 × 2/9 = 6/90 = 1/15

(ii) the first is red and the second is green
P = 3/10 × 2/9 = 6/90 = 1/15

(iii) the two balls are of the same colour
Red-red: 3/10 × 2/9 = 6/90
Green-green: 2/10 × 1/9 = 2/90
Blue-blue: 5/10 × 4/9 = 20/90
Total = (6 + 2 + 20)/90 = 28/90 = 14/45

Probability = 14/45
8. (a) (i) Evaluate the value of tan 15° from sine and cosine of 45° and 30°

Use identity:
$\tan(A - B) = (\tan A - \tan B) / (1 + \tan A \tan B)$
$\tan 15° = \tan(45° - 30°)$
$\tan 45° = 1$, $\tan 30° = 1/\sqrt{3}$
So:
$\tan 15° = (1 - 1/\sqrt{3}) / (1 + (1)(1/\sqrt{3}))$
$= (\sqrt{3} - 1) / (\sqrt{3} + 1)$
Multiply numerator and denominator by $(\sqrt{3} - 1)$:
$= (\sqrt{3} - 1)^2 / [(\sqrt{3} + 1)(\sqrt{3} - 1)] = (3 - 2\sqrt{3} + 1)/(3 - 1) = (4 - 2\sqrt{3})/2 = 2 - \sqrt{3}$

(ii) Given that sin A = 3/5 and cos B = 15/17 where A and B are in the second quadrants respectively, find the exact value of sin(A + B)

$\sin A = 3/5 \rightarrow$ A in second quadrant $\rightarrow \cos A = -\sqrt{(1 - 9/25)} = -4/5$
$\cos B = 15/17 \rightarrow$ B in second quadrant $\rightarrow \sin B = \sqrt{(1 - 225/289)} = \sqrt{64/289} = 8/17$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$
$= (3/5)(15/17) + (-4/5)(8/17) = 45/85 - 32/85 = 13/85$

(b) Prove that $\sin 2A = (2 \tan A)/(1 + \tan^2 A)$

LHS: $\sin 2A = 2 \sin A \cos A$
Let $\tan A = \sin A / \cos A$
Then:
RHS $= (2 \sin A / \cos A) / (1 + \sin^2 A / \cos^2 A)$
$= (2 \sin A / \cos A) / ((\cos^2 A + \sin^2 A) / \cos^2 A) = (2 \sin A \cos A) / (\cos^2 A + \sin^2 A)$
But $\sin^2 A + \cos^2 A = 1$

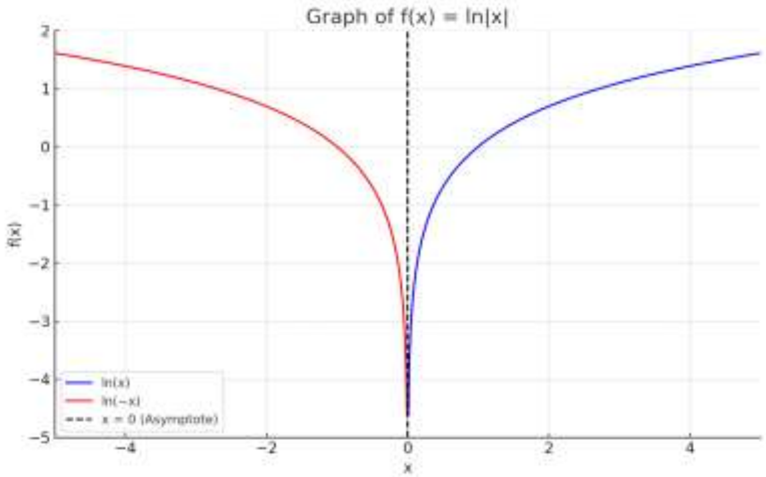
So RHS = 2 sin A cos A = LHS

Identity is proved.

9. (a) Sketch the graphs of $f(x) = \ln|x|$ for $x \in \mathbb{R}$ and hence state its domain and range.

The graph consists of:
- $\ln(x)$ for $x > 0$ (increasing curve)
- $\ln(-x)$ for $x < 0$ (mirror image across y-axis)

Vertical asymptote at $x = 0$



Domain: $x \in \mathbb{R}, x \neq 0$
Range: $(-\infty, \infty)$

(b) The amount (A) of the radioactive isotope Carbon-14 at any time t is given by the formula $A(t) = A_0e^{kt}$ where A_0 is the initial amount of the element. If the half-life of the radioactive isotope Carbon-14 is about 5730 years:

(i) Express the amount of Carbon-14 left from an initial A_0 milligrams as a function of time t in years.

When $t = 5730$, $A(t) = A_0/2$
$A_0/2 = A_0 e^{(k \times 5730)}$
Divide both sides by A_0: $1/2 = e^{(5730k)}$
Take ln: $\ln(1/2) = 5730k \rightarrow k = \ln(0.5)/5730 = -0.000121$

So: $A(t) = A_0 e^{(-0.000121t)}$

(ii) What percentage of the original amount of Carbon-14 left after 20000 years?

$A(20000) = A_0 e^{(-0.000121 \times 20000)} = A_0 e^{(-2.42)}$
$= A_0 \times 0.0889$
So 8.89% of the original amount remains

Percentage ≈ 8.89%

10. An aircraft has 600 m^2 of cabin space and can carry 5000 kg of luggage. An economy class passenger gets 3 m^2 of space and is allowed to travel with 20 kg of luggage. The first class passenger gets 4 m^2 of space and is allowed to have 50 kg of luggage in the aircraft. In the aircraft, there is space for at least 50 economy class passengers. The profit per flight for the economy and first class passengers are 40,000/- and 100,000/- respectively.

Let:
x = number of economy class passengers
y = number of first class passengers

(a) Write down all the constraints.

Cabin space constraint:
$3x + 4y \leq 600$

Luggage constraint:
$20x + 50y \leq 5000$

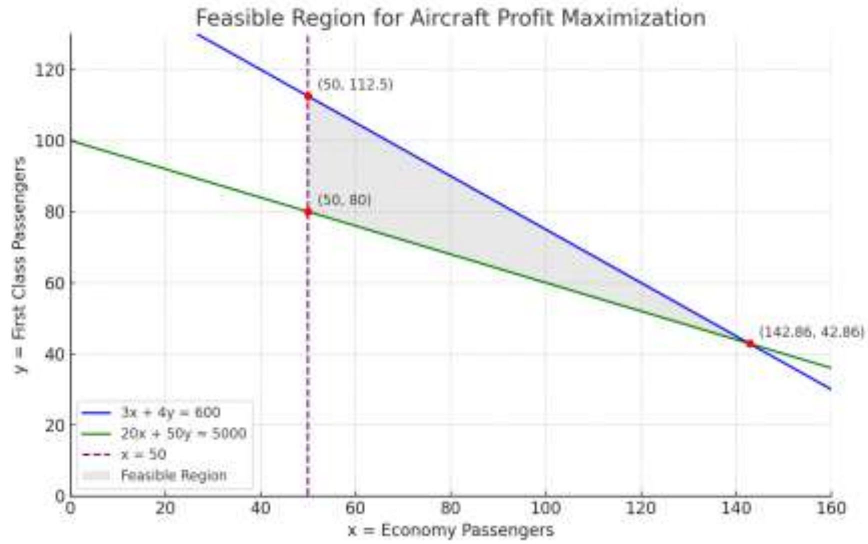
Minimum economy passengers:
$x \geq 50$

Also, $x \geq 0$ and $y \geq 0$

Objective function (profit):
$P = 40000x + 100000y$

(b) Use graphical method to find the number of economy passengers and first class passengers which will give the maximum profit per flight.

To solve, we graph:
$3x + 4y = 600$
$\rightarrow y = (600 - 3x)/4$
$20x + 50y = 5000$
$\rightarrow y = (5000 - 20x)/50 = 100 - 0.4x$



Now compute profit at the points within feasible region:
- (50, 80): P = 40000×50 + 100000×80 = 2,000,000 + 8,000,000 = 10,000,000
- (50, 112.5): P = 40000×50 + 100000×112.5 = 2,000,000 + 11,250,000 = 13,250,000
- (142.86, 42.86): P = 40000×142.86 + 100000×42.86 ≈ 5,714,400 + 4,286,000 = 10,000,400

Maximum profit is at (50, 112.5)

Number of economy passengers = 50, Number of first class passengers = 112.5
Maximum profit = Tsh 13,250,000