THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL

CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

042 ADITIONAL MATHEMATICS

(For Both School and Private Candidates)

Time: 3 Hours **ANSWERS** **Year: 2001**

Instructions

1. This paper consists of SIXTEEN questions.
2. Answer all questions in section A and any FOUR questions in section B.

1. (a) The functions f and g are defined by
f: $x \rightarrow 5x + 4$
g: $x \rightarrow 6x - k$
where $x \in \mathbb{R}$ and k is a constant.
Find the value of k for which $fog(x) = gof(x)$.

$fog(x) = f(g(x)) = f(6x - k) = 5(6x - k) + 4 = 30x - 5k + 4$
$gof(x) = g(f(x)) = g(5x + 4) = 6(5x + 4) - k = 30x + 24 - k$

Now equate:
$30x - 5k + 4 = 30x + 24 - k$
Cancel 30x:
$-5k + 4 = 24 - k$
Rearrange: $-5k + k = 24 - 4 \rightarrow -4k = 20 \rightarrow k = -5$

(b) The expression $3x^3 + 2x^2 - bx + a$ is divisible by $(x - 1)$ and leaves a remainder of 9 when divided by $(x + 1)$. Find the values of a and b.

Let $f(x) = 3x^3 + 2x^2 - bx + a$
If $(x - 1)$ is a factor, then $f(1) = 0$
$f(1) = 3 + 2 - b + a = 0 \rightarrow a - b = -5$ ...(i)
Also, $f(-1) = 9$
$f(-1) = -3 + 2 + b + a = 9 \rightarrow a + b = 10$ ...(ii)

Add (i) and (ii):
$a - b + a + b = -5 + 10 \rightarrow 2a = 5 \rightarrow a = 2.5$
Then $b = 10 - 2.5 = 7.5$

2. (a) Prove that $(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta) = \sin^3\theta + \cos^3\theta$.

Left-hand side:
$(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$
$= \sin\theta(1 - \sin\theta\cos\theta) + \cos\theta(1 - \sin\theta\cos\theta)$
$= \sin\theta - \sin^2\theta\cos\theta + \cos\theta - \sin\theta\cos^2\theta$
$= \sin\theta + \cos\theta - \sin^2\theta\cos\theta - \sin\theta\cos^2\theta$
Group terms:
$= \sin\theta + \cos\theta - \sin\theta\cos\theta(\sin\theta + \cos\theta)$
$= (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$ — confirmed equal to LHS
Now RHS:
$\sin^3\theta + \cos^3\theta$
Use identity: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
$\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta)$
$= (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$ — same as LHS

So identity is proven.

(b) Find the value of $\sin^2 A \operatorname{cosec}^2(\pi/2 - A) - \cot^2(\pi/2 - A)\cos^2 A$

We use identities:
$\operatorname{cosec}(\pi/2 - A) = \sec A \rightarrow \operatorname{cosec}^2(\pi/2 - A) = \sec^2 A$
$\cot(\pi/2 - A) = \tan A \rightarrow \cot^2(\pi/2 - A) = \tan^2 A$
Now expression becomes:
$\sin^2 A \times \sec^2 A - \tan^2 A \times \cos^2 A$
$\sin^2 A \times (1/\cos^2 A) - (\sin^2 A/\cos^2 A) \times \cos^2 A$
$= \sin^2 A/\cos^2 A - \sin^2 A = \tan^2 A - \sin^2 A$

Final simplified result: $\tan^2 A - \sin^2 A$

3. If u = 2i - j, v = 6i - 3j

(i) Find u + v
u + v = (2 + 6)i + (-1 - 3)j = 8i - 4j

(ii) Show that |u + v| = |u| + |v|
$|u + v| = \sqrt{(8^2 + (-4)^2)} = \sqrt{(64 + 16)} = \sqrt{80}$
$|u| = \sqrt{(4 + 1)} = \sqrt{5}$
$|v| = \sqrt{(36 + 9)} = \sqrt{45}$
$|u| + |v| = \sqrt{5} + \sqrt{45}$
Since $\sqrt{80} \neq \sqrt{5} + \sqrt{45} \rightarrow$ statement is false

4. (a) In a triangle OAB, OA = a and OB = b. Given that P and Q are the midpoints of OA and OB respectively, express vector PQ and AB in terms of a and b.

OP = ½a, OQ = ½b
PQ = OQ - OP = ½b - ½a = ½(b - a)
AB = b - a
So PQ = ½ AB
Relationship: PQ is parallel to AB and half its length

(b) State the geometrical relationship between PQ and AB.
They are parallel and PQ = ½ AB

5. Calculate the median and upper quartile of the distribution.

Mass (g)	Frequency	Cumulative Freq
75–79	7	7

80–84	30	37
85–89	66	103
90–94	57	160
95–99	27	187
100–104	13	200

Median class = $(200/2 = 100) \rightarrow$ class 85–89
Median = $L + [(n/2 - F)/f] \times h$
$L = 84.5$, $F = 37$, $f = 66$, $h = 5$
Median = $84.5 + [(100 - 37)/66] \times 5 = 84.5 + (63/66)\times5 \approx 84.5 + 4.77 = 89.27$g

Upper quartile: $\frac{3}{4} \times 200 = 150 \rightarrow$ class 90–94
$L = 89.5$, $F = 103$, $f = 57$
$Q3 = 89.5 + [(150 - 103)/57] \times 5 = 89.5 + (47/57)\times5 \approx 89.5 + 4.12 = 93.62$g

6. A social committee is to transport 20 boys and 32 girls to a place for a picnic.
The committee can hire either a taxi (2 boys + 1 girl) or a mini bus (2 boys + 4 girls).
Costs: taxi = 4000, mini bus = 3000
Find the cheapest means of transport.

Let x = no. of taxis, y = no. of mini buses
$2x + 2y = 20 \rightarrow x + y = 10$
$x + 4y = 32$
From above: $y = 10 - x$
Substitute: $x + 4(10 - x) = 32 \rightarrow x + 40 - 4x = 32 \rightarrow -3x = -8 \rightarrow x = 8/3$

x not whole number, try trial:
$x = 4 \rightarrow y = 6 \rightarrow 2x + 2y = 8 + 12 = 20$ boys, $x + 4y = 4 + 24 = 28$ girls
Not 32
Try $x = 2 \rightarrow y = 8$
$2x + 2y = 4 + 16 = 20$ boys
$x + 4y = 2 + 32 = 34$ girls $\rightarrow$ exceeds
Try $x = 3$, $y = 7 \rightarrow 6 + 14 = 20$ boys, $3 + 28 = 31$ girls $\rightarrow$ too small
Try $x = 4$, $y = 6 \rightarrow 8 + 12 = 20$ boys, $4 + 24 = 28$ girls
Only $x = 6$, $y = 5 \rightarrow 12 + 10 = 22$ boys, $6 + 20 = 26$ girls
Try $x = 5$, $y = 5 \rightarrow 10 + 10 = 20$ boys, $5 + 20 = 25$ girls
Only valid: $x = 4$, $y = 7 \rightarrow 8 + 14 = 22$ boys, $4 + 28 = 32$ girls
Finally: $x = 2$, $y = 8 \rightarrow$ costs $= 2\times4000 + 8\times3000 = 8000 + 24000 = 32000$
Cheapest cost = 32000

7. (a) Differentiate with respect to x:

(i) $y = \sin x / (1 + \tan x)$

Use quotient rule:
$dy/dx = [\cos x(1 + \tan x) - \sin x(\sec^2 x)] / (1 + \tan x)^2$

(ii) $y = \sqrt{x^2 + 2x}$
Let $u = x^2 + 2x$
$dy/dx = \frac{1}{2}(x^2 + 2x)^{-1/2} \times (2x + 2) = (x + 1)/\sqrt{x^2 + 2x}$

(b) Evaluate $\int_0^{\pi/2} (2\cos^2\theta + 3\sin^2\theta)\, d\theta$
$= \int_0^{\pi/2} [2(1 + \cos 2\theta)/2 + 3(1 - \cos 2\theta)/2]\, d\theta$
$= \int_0^{\pi/2} [2/2 + 2\cos 2\theta/2 + 3/2 - 3\cos 2\theta/2]$
$= \int_0^{\pi/2} [(5/2) - (\cos 2\theta)/2]$
$= 5\theta/2 - \sin 2\theta/4$ from 0 to $\pi/2$
$= 5\pi/4 - 0 = 5\pi/4$

8. (a) Show how the biconditional $p \leftrightarrow q$ can be written in terms of original three connectives $\vee$, $\wedge$, $\neg$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

(b) Using truth table verify that:

(i) $\neg(p \longrightarrow q) \equiv p \wedge \neg q$
(ii) $\neg(p \leftrightarrow q) \equiv p \oplus q$ (exclusive or)

Use full 2-variable truth table to verify both expressions

9. (a) $\int e^x \cos 3x\, dx$
Use integration by parts twice or known result:
$= (e^x/10)(\cos 3x + 3 \sin 3x) + C$

(b) Find p and q such that
$\int_0^x (pt - q)\, dt = 9x^2 + 9x$
Integrate: $(pt - q) \longrightarrow p \times t^2/2 - qt$
Apply limits:
$p x^2/2 - qx = 9x^2 + 9x$
So: $p/2 = 9 \longrightarrow p = 18$
$-q = 9 \longrightarrow q = -9$

10. Solve the following system of simultaneous equations:
$2x - 5y + 2z = 14$
$9x + 3y - 4z = 13$
$7x + 3y - 2z = 3$
Given,
$2x - 5y + 2z = 14$ ………………… (1)

$9x + 3y - 4z = 13$ ……………… (2)
$7x + 3y - 2z = 3$ ………………… (3)

Step 1: Eliminate y from equations (2) and (3)

We can subtract (2) - (3) directly because both have 3y:

(2) - (3):
$(9x - 7x) + (3y - 3y) + (-4z + 2z) = 13 - 3$
$\rightarrow 2x - 2z = 10$ …………………… (4)

Step 2: Eliminate y from equations (1) and (2)

We want to eliminate y, so we find LCM of coefficients of y in (1) and (2): LCM of 5 and 3 is 15

Multiply (1) by 3:
$3(2x - 5y + 2z) = 6x - 15y + 6z = 42$ ……… (5)

Multiply (2) by 5:
$5(9x + 3y - 4z) = 45x + 15y - 20z = 65$ ……… (6)

Now add (5) and (6):
$(6x + 45x) + (-15y + 15y) + (6z - 20z) = 42 + 65$
$51x - 14z = 107$ …………………………… (7)

Step 3: Use equations (4) and (7) to eliminate another variable

We now solve equations (4) and (7):

(4): $2x - 2z = 10$
(7): $51x - 14z = 107$

Multiply (4) by 7 to make coefficients of z equal ($2\times7 = 14$):
$7(2x - 2z) = 14x - 14z = 70$ ……………………… (8)

Now subtract (8) from (7):
$(51x - 14z) - (14x - 14z) = 107 - 70$
$51x - 14x = 37x$
$\rightarrow 37x = 37$
$\rightarrow x = 1$

Step 4: Substitute x = 1 into equation (4) to find z

From (4):
$2x - 2z = 10$
$2(1) - 2z = 10 \rightarrow 2 - 2z = 10 \rightarrow -2z = 8 \rightarrow z = -4$

Step 5: Substitute $x = 1$, $z = -4$ into equation (1) to find y

From (1):
$2x - 5y + 2z = 14$
$2(1) - 5y + 2(-4) = 14$
$2 - 5y - 8 = 14 \rightarrow -5y - 6 = 14 \rightarrow -5y = 20 \rightarrow y = -4$

Final solution:
$x = 1, y = -4, z = -4$

11. (a) Find the 2×2 matrix that will transform the point (1, 2) to (3, 3) and the point (−1, 1) to (−3, 3)

Let the matrix be:
M = [a b]
[c d]

We want:
[a b][1] = [3] and [a b][−1] = [−3]
[c d][2]   [3]   [c d][1]   [3]

So the system:
$a(1) + b(2) = 3 \rightarrow a + 2b = 3 \ldots (1)$
$c(1) + d(2) = 3 \rightarrow c + 2d = 3 \ldots (2)$
$a(-1) + b(1) = -3 \rightarrow -a + b = -3 \ldots (3)$
$c(-1) + d(1) = 3 \rightarrow -c + d = 3 \ldots (4)$

From (1) and (3):
$a + 2b = 3$
$-a + b = -3 \rightarrow$ add $\rightarrow 3b = 0 \rightarrow b = 0$
Then $a = 3$

From (2) and (4):
$c + 2d = 3$
$-c + d = 3 \rightarrow$ add $\rightarrow 3d = 6 \rightarrow d = 2$
Then $c = 3 - 2d = 3 - 4 = -1$

So matrix is:
[3 0]
[−1 2]

(b) All the points on the line $y = 2x - 3$ have been transformed by the matrix
$$\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

Find the equation of the image line.

Let (x, y) be on the line $y = 2x - 3$
Then image:
$x' = 2x + y$
$y' = 3x - y$
But $y = 2x - 3 \rightarrow$ substitute:
$x' = 2x + (2x - 3) = 4x - 3$
$y' = 3x - (2x - 3) = x + 3$
Eliminate x:
From $x' = 4x - 3 \rightarrow 4x = x' + 3 \rightarrow x = (x' + 3)/4$
Then $y' = (x' + 3)/4 + 3 = (x' + 3 + 12)/4 = (x' + 15)/4$
Equation of image line: $y' = (1/4)x' + 15/4$

12. Find the equation of the circle which circumscribes the triangle with vertices (1, 0), (2, 1), and (0, 2)

Let A(1, 0), B(2, 1), C(0, 2)
Find perpendicular bisectors of AB and BC and find intersection (center)

Midpoint AB: $((1+2)/2, (0+1)/2) = (1.5, 0.5)$, slope AB $= (1-0)/(2-1) = 1$
Perpendicular slope $= -1$, equation: $y - 0.5 = -1(x - 1.5)$

$\rightarrow y = -x + 2$

Midpoint BC $= ((2+0)/2, (1+2)/2) = (1, 1.5)$, slope BC $= (2-1)/(0-2) = -1/2$
Perpendicular slope $= 2$, equation: $y - 1.5 = 2(x - 1)$

$\rightarrow y = 2x - 0.5$

Solve: $-x + 2 = 2x - 0.5 \rightarrow 3x = 2.5 \rightarrow x = 5/6$
Then $y = -5/6 + 2 = 7/6$

Center $= (5/6, 7/6)$, use A to find radius:
$r^2 = (1 - 5/6)^2 + (0 - 7/6)^2 = (1/6)^2 + (-7/6)^2 = 1/36 + 49/36 = 50/36$
So equation:
$(x - 5/6)^2 + (y - 7/6)^2 = 50/36$

13. (a) A company wishes to pack food in closed cylindrical tins of volume 128π cm³.

Find dimensions of minimum area.

$V = \pi r^2 h = 128\pi \rightarrow h = 128/r^2$
Area $A = 2\pi r^2 + 2\pi rh$
Substitute $h = 128/r^2$:
$A = 2\pi r^2 + 2\pi r(128/r^2) = 2\pi r^2 + 256\pi/r$

Differentiate A:
$dA/dr = 4\pi r - 256\pi/r^2$
Set to 0: $4\pi r = 256\pi/r^2 \rightarrow 4r^3 = 256 \rightarrow r^3 = 64 \rightarrow r = 4$
Then $h = 128/16 = 8$
Dimensions: radius = 4 cm, height = 8 cm

(b) Volume generated by rotating region between $y = 5$ and $y = x^2 + 1$ about $y = 5$

$y = 5 - (x^2 + 1) = 4 - x^2$ is radius
$V = \pi \int$ from $x = -2$ to 2 of $(4 - x^2)^2\, dx$
$= \pi \int_{-2}^{2} (16 - 8x^2 + x^4)\, dx = \pi[16x - 8x^3/3 + x^5/5]_{-2}^{2}$
Compute definite integral:
At $x = 2$: $32 - 64/3 + 32/5 = 160/5 - 64/3 + 32/5$
$= (160 + 32)/5 = 192/5$
$\rightarrow 192/5 - 64/3$
Common denom = 15 $\rightarrow (576 - 320)/15 = 256/15$
Double it: final $V = 2 \times 256/15 \times \pi = 512\pi/15\ cm^3$

14. (a) $P(A) = 5/8$, $P(B') = 3/7 \rightarrow P(B) = 4/7$
$P(A \cap B) = P(A) \times P(B) = 5/8 \times 4/7 = 20/56 = 5/14$

(b) Class of 30 boys
15 have bicycles
10 have motorbikes
4 have both
So:
Bicycle only = 15 − 4 = 11
Motorbike only = 10 − 4 = 6
None = 30 − (11 + 6 + 4) = 9

(i) Neither = 9/30 = 3/10
(ii) Bicycle but no motorbike = 11/30

15. (a) $z = \frac{1}{2} - \frac{\sqrt{3}}{2} i$
Modulus $= \sqrt{[(\frac{1}{2})^2 + (\sqrt{3}/2)^2]} = \sqrt{(1/4 + 3/4)} = \sqrt{1} = 1$
Argument $= \tan^{-1}(-\sqrt{3}) = -\pi/3$

(b) $z = 1 + i$
$z^3 = (1 + i)^3 = 1 + 3i + 3i^2 + i^3 = 1 + 3i - 3 - i = -2 + 2i$
Hence $z^3 = -2 + 2i$

$p/(1 + z) = q/(1 + z^3) = 2i$
$1 + z = 1 + 1 + i = 2 + i$
$1 + z^3 = 1 - 2 + 2i = -1 + 2i$
So:
$p/(2 + i) = 2i \rightarrow p = 2i(2 + i) = 4i + 2i^2 = 4i - 2 = -2 + 4i$
$q/(-1 + 2i) = 2i \rightarrow q = 2i(-1 + 2i) = -2i + 4i^2 = -2i - 4 = -4 - 2i$

(c) $2z - 4\bar{z} = 3 - 6i$
Let $z = x + iy \rightarrow \bar{z} = x - iy$
$2(x + iy) - 4(x - iy) = 3 - 6i$
$2x + 2iy - 4x + 4iy = 3 - 6i$
$-2x + 6iy = 3 - 6i$
Compare real and imaginary:
$-2x = 3 \rightarrow x = -1.5$
$6iy = -6i \rightarrow y = -1$
So $z = -1.5 - i$

16. Solve $x^3 - x - 6 = 0$ between 1 and 2 using Newton-Raphson, starting $x_0 = 1.6$

$f(x) = x^3 - x - 6$
$f'(x) = 3x^2 - 1$

$x_1 = x_0 - f(x_0)/f'(x_0)$
$f(1.6) = 4.096 - 1.6 - 6 = -3.504$
$f'(1.6) = 3(2.56) - 1 = 6.68$
$x_1 = 1.6 - (-3.504)/6.68 = 1.6 + 0.524 = 2.124$

$x_2 = 2.124 - f(2.124)/f'(2.124)$
$f(2.124) = 9.58 - 2.124 - 6 = 1.456$
$f' = 3(4.512) - 1 = 12.54$
$x_2 = 2.124 - (1.456)/12.54 \approx 2.008$

Better root ≈ 2.01 to 2 d.p.