THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL OF TANZANIA

CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

042 ADDITIONAL MATHEMATICS

Time: 3hour **SOLUTIONS** **Year: 2020**

Instructions

1. This paper consists of sections A and B with a total of fourteen (14) questions.
2. Answer all questions.
3. Section A carries sixty (60) marks and section B carries forty (40) marks.
4. All necessary working and answers for each question attempted must be shown clearly.
5. NECTA Mathematical tables and non-programmable calculators may be used.
6. All communication devices and any unauthorised materials are not allowed in the examination room.
7. Write your Examination Number on every page of your answer booklet(s).

1. (a) Given that y is proportional to x and y = 4 when x = 2:

(i) Formulate an equation that relates x and y.

Since y is proportional to x,

$y \propto x$

$y = kx$

Given y = 4 when x = 2,

$4 = k \times 2$

$k = 2$

Therefore, the required equation is:

$y = 2x$

(ii) Sketch a graph showing the relationship of x and y.

The graph of y = 2x is a straight line passing through the origin.

It passes through points such as (0,0), (1,2), (2,4), and (3,6).

The line has a constant positive gradient of 2.

(b) The electrical resistance (R) of a copper wire of a circular cross section area varies directly to the length (l) and inversely to the square of the radius (r). Two wires have equal resistance and one is four times as long as the other. Find the ratio of their radii.

$R \propto l / r^2$

$R = k\,l / r^2$

For two wires with equal resistance:

$k\,l_1 / r_1^2 = k\,l_2 / r_2^2$

Cancel k:

$l_1 / r_1^2 = l_2 / r_2^2$

Given one wire is four times as long as the other,

let $l_1 = 4l_2$

Substitute:

$4l_2 / r_1^2 = l_2 / r_2^2$

Cancel l_2:

$4 / r_1^2 = 1 / r_2^2$

Cross multiply:

$r_1^2 = 4r_2^2$

Take square root:

$r_1 = 2r_2$

Therefore, the ratio of their radii is:

$r_1 : r_2 = 2 : 1$

2. The following frequency distribution table shows the amount of money spent by each person in a certain shop.

Amount of money spent (Shs)	0–500	500–1000	1000–1500	1500–2000	2000–2500	2500–3000
Frequency:	6	16	27	18	9	2

(a) Determine the lower and upper quartiles correct to two decimal places.

Total frequency, N =

6 + 16 + 27 + 18 + 9 + 2 = 78

Lower quartile position:

Q1 = N / 4 = 78 / 4 = 19.5

Upper quartile position:

Q3 = 3N / 4 = 58.5

Cumulative frequencies:

6, 22, 49, 67, 76, 78

Q1 lies in the class 500–1000.

Use class boundary 499.5–999.5, class width h = 500.

$Q1 = 499.5 + [(19.5 - 6) / 16] \times 500$

$Q1 = 499.5 + (13.5 / 16) \times 500$

$Q1 = 499.5 + 421.88$

$Q1 = 921.38$

Q3 lies in the class 1500–2000.

Lower boundary = 1499.5, frequency = 18, cumulative before = 49

$Q3 = 1499.5 + [(58.5 - 49) / 18] \times 500$

$Q3 = 1499.5 + (9.5 / 18) \times 500$

$Q3 = 1499.5 + 263.89$

$Q3 = 1763.39$

(b) Find the variance and standard deviation correct to two decimal places.

Class midpoints x:

250, 750, 1250, 1750, 2250, 2750

Compute fx and fx^2:

$\Sigma fx =$

$6(250) + 16(750) + 27(1250) + 18(1750) + 9(2250) + 2(2750)$

$= 1500 + 12000 + 33750 + 31500 + 20250 + 5500$

$= 104500$

$\Sigma fx^2 =$

$6(250^2) + 16(750^2) + 27(1250^2) + 18(1750^2) + 9(2250^2) + 2(2750^2)$

$= 375000 + 9000000 + 42187500 + 55125000 + 45562500 + 15125000$

$= 167,375,000$

Mean:

$\bar{x} = \Sigma fx / \Sigma f$

$= 104500 / 78$

$= 1339.74$

Variance:

$\sigma^2 = (\Sigma fx^2 / \Sigma f) - \bar{x}^2$

$\sigma^2 = (167375000 / 78) - (1339.74)^2$

$\sigma^2 = 2145833.33 - 1794893.43$

$\sigma^2 = 350939.90$

Standard deviation:

$\sigma = \sqrt{350939.90}$

$\boldsymbol{\sigma = 592.39}$

3. (a) Determine the perpendicular distance of the point $(2, -1)$ from the line $3x + 4y - 6 = 0$.

   Distance formula:

   $d = |ax + by + c| / \sqrt{(a^2 + b^2)}$

   Substitute values:

   $d = |3(2) + 4(-1) - 6| / \sqrt{(3^2 + 4^2)}$

   $d = |6 - 4 - 6| / \sqrt{25}$

   $d = 4 / 5$

   Distance = 0.8 units.

(b) Find the equation of a circle whose diameter has the coordinates (1,2) and (−1,3) as its end points.

Centre = midpoint:

$((1 - 1)/2, (2 + 3)/2) = (0, 5/2)$

Radius = half the distance between endpoints:

Distance $= \sqrt{[(1 + 1)^2 + (2 - 3)^2]}$

$= \sqrt{(4 + 1)}$

$= \sqrt{5}$

Radius $= \sqrt{5} / 2$

Equation of the circle:

$(x - 0)^2 + (y - 5/2)^2 = (\sqrt{5} / 2)^2$

$\mathbf{x^2 + (y - 5/2)^2 = 5/4}$

4. The point P(x, y) moves in a plane in such a way that it is always 5 units from the fixed point O(2,3).

(a) Sketch the locus of the point P(x, y) in the xy-plane.

The locus is a circle with centre at (2,3) and radius 5.

(b) Find the equation of the locus of the point P(x, y) in the form
$x^2 + y^2 - 2fx - 2gy + c = 0$.

Using the distance formula:

$(x - 2)^2 + (y - 3)^2 = 25$

Expand:
$x^2 - 4x + 4 + y^2 - 6y + 9 = 25$

Simplify:
$\mathbf{x^2 + y^2 - 4x - 6y - 12 = 0}$

5. (a) Given the formula
$D / d = \sqrt{(f + p) / (f - p)}$, express p in terms of D, d and f and then find the value of p when D = 2, f = 5 and d = 1.

Square both sides:

$(D / d)^2 = (f + p) / (f - p)$

Cross multiply:
$D^2(f - p) = d^2(f + p)$

Expand:
$D^2f - D^2p = d^2f + d^2p$

Group terms:

$D^2f - d^2f = D^2p + d^2p$

Factor:

$f(D^2 - d^2) = p(D^2 + d^2)$

Therefore:

$p = f(D^2 - d^2) / (D^2 + d^2)$

Substitute values:

$p = 5(4 - 1) / (4 + 1)$

$p = 15 / 5$

$\mathbf{p = 3}$

(b) The perimeter of a rectangle is 46 cm. If one of the two adjacent sides is 7 cm longer than the other, find the dimensions of the rectangle.

Let the shorter side be x cm.

Longer side = $x + 7$

Perimeter:

$2(x + x + 7) = 46$

$2(2x + 7) = 46$

$4x + 14 = 46$

$4x = 32$

$x = 8$

So the longer side = 15

Dimensions of the rectangle are:

8 cm by 15 cm

6. (a) Use the following polygon to answer the given questions.

(i) Write the name of the polygon.

The polygon shown has five sides.

Therefore, it is called a pentagon.

(ii) How many triangles does the polygon have?

Number of triangles in a polygon = n − 2

Here, n = 5

Number of triangles = 5 − 2

= 3

(iii) Compute the sum of all interior angles of the polygon.

Sum of interior angles = (n − 2) × 180

For n = 5,

Sum = (5 − 2) × 180

= 3 × 180

= 540 degrees

(iv) Find the size of each interior angle in the polygon.

Since the polygon shown is regular,

Each interior angle = 540 / 5

= 108 degrees

(b) Use diagrams to find the number of axes of symmetry in each of the following.

(i) Cube

A cube has symmetry through planes passing through opposite faces, edges, and diagonals.

Number of axes of symmetry = 13

(ii) Rectangle

A rectangle has symmetry along the vertical and horizontal lines passing through its center.

Number of axes of symmetry = 2

(iii) Sphere

A sphere is symmetrical about every diameter.

Number of axes of symmetry = infinitely many

7. (a) Prove that

$2 \sin \theta + \sin 2\theta/1 - \cos \theta = \sin \theta/1 - \cos \theta$

Start with the left hand side.

$2 \sin \theta + \sin 2\theta$

Use identity $\sin 2\theta = 2 \sin \theta \cos \theta$

$= 2 \sin \theta + 2 \sin \theta \cos \theta$

$= 2 \sin \theta(1 + \cos \theta)$

Now the denominator:

$1 - \cos 2\theta$

Use identity $\cos 2\theta = 2 \cos^2\theta - 1$

$1 - (2\cos^2\theta - 1)$

$= 2 - 2\cos^2\theta$

$= 2\sin^2\theta$

So the expression becomes:

$[2\sin\theta(1 + \cos\theta)] / [2\sin^2\theta]$

Cancel $2\sin\theta$:

$= (1 + \cos\theta) / \sin\theta$

Multiply numerator and denominator by $(1 - \cos\theta)$:

$= (1 - \cos^2\theta) / [\sin\theta(1 - \cos\theta)]$

$= \sin^2\theta / [\sin\theta(1 - \cos\theta)]$

$= \sin\theta / (1 - \cos\theta)$

Hence proved.

(b) Without using mathematical tables or calculators, find the value of tan 105° in surd form.

$\tan 105° = \tan(60° + 45°)$

Using the formula:

$\tan(A + B) = (\tan A + \tan B) / (1 - \tan A \tan B)$

$\tan 60° = \sqrt{3}$

$\tan 45° = 1$

$\tan 105° = (\sqrt{3} + 1) / (1 - \sqrt{3})$

Rationalize the denominator:

$= (\sqrt{3} + 1)(1 + \sqrt{3}) / [(1 - \sqrt{3})(1 + \sqrt{3})]$

$= ((\sqrt{3} + 1)(1 + \sqrt{3})) / (1 - 3)$

$\mathbf{= (4 + 2\sqrt{3}) / (-2)}$

$\mathbf{= -2 - \sqrt{3}}$

8. (a) Applying rules for divisibility, show that 637 is divisible by 13.

$637 \div 13 = 49$

Since the division gives a whole number,

637 is divisible by 13.

(b) Find the sum of the first n terms of the series

$1 + 3 + 5 + \ldots + (2n - 1)$

and then show that

$1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = (n + 1)^2$.

Sum of first n odd numbers:

$S_n = n^2$

Now add the next term $(2n + 1)$:

$S_n + (2n + 1)$

$= n^2 + 2n + 1$

$= (n + 1)^2$

Hence shown that:

$1 + 3 + 5 + \dots + (2n - 1) + (2n + 1) = (n + 1)^2$.

9. (a) (i) Write the simplified compound statement for S(p, q, r) having the following truth table.

From the truth table, S(p, q, r) is true only in the following cases:

p = T, q = T, r = T

p = T, q = F, r = T

Factor the common true conditions.

In both cases, p = T and r = T.

The value of q does not affect the result.

Therefore,

$S(p, q, r) = p \wedge r$

This is the simplified compound statement.

(ii) Draw a simple network for S(p, q, r).

Since $S(p, q, r) = p \wedge r$, the network consists of two switches p and r connected in series. Current flows only when both p and r are closed.

(b) By using the laws of algebra of propositions, simplify $p \vee (p \wedge q)$.

Start with the given expression:

$p \vee (p \wedge q)$

Apply the absorption law:

$p \vee (p \wedge q) = p$

10. (a) Simplify $A \cap (A' \cap B)$ using the laws of set operations.

Start with the given expression:

$A \cap (A' \cap B)$

Apply the associative law:

$= (A \cap A') \cap B$

But

$A \cap A' = \emptyset$

So,

$(A \cap A') \cap B = \emptyset \cap B = \emptyset$

(b) In a meeting of 30 people, 15 are farmers, 20 are teachers and 8 are both teachers and farmers. By using a Venn diagram, find the number of people who are neither farmers nor teachers.

Let

F = set of farmers

T = set of teachers

Given:

$n(F) = 15$

$n(T) = 20$

$n(F \cap T) = 8$

Total = 30

Find number in $F \cup T$:

$n(F \cup T) = n(F) + n(T) - n(F \cap T)$

$= 15 + 20 - 8$

$= 27$

People who are neither farmers nor teachers:

$=$ Total $- n(F \cup T)$

$= 30 - 27$

$= 3$

Number of people who are neither farmers nor teachers = 3

11. (a) When the equation $ax^3 - x^2 + 7x + c = 0$ is divided by $x - 1$ and $x + 1$, the remainders are -4 and 6 respectively. Find the values of a and c.

By the remainder theorem:

For $x - 1$:

$f(1) = -4$

Substitute $x = 1$:

$a(1)^3 - (1)^2 + 7(1) + c = -4$

$a - 1 + 7 + c = -4$

$a + c + 6 = -4$

$a + c = -10$ …… (1)

For $x + 1$:

$f(-1) = 6$

Substitute $x = -1$:

$a(-1)^3 - (-1)^2 + 7(-1) + c = 6$

$-a - 1 - 7 + c = 6$

$-a + c - 8 = 6$

$-a + c = 14$ …… (2)

Solve (1) and (2):

From (1): $c = -10 - a$

Substitute into (2):

$-a + (-10 - a) = 14$

$-2a - 10 = 14$

$-2a = 24$

$a = -12$

Then

$c = -10 - (-12)$

$c = 2$

Therefore,

$a = -12$ and $c = 2$

(b) Given that $f(x) = (x + 1) / (x^2 + x - 2)$, find the intercepts and asymptotes of $f(x)$.

Factor the denominator:

$x^2 + x - 2 = (x + 2)(x - 1)$

Intercepts.

x intercept:

Numerator = 0

$x + 1 = 0$

$x = -1$

x intercept is $(-1, 0)$

y intercept:

$x = 0$

$f(0) = (0 + 1) / (-2)$

$= -1/2$

y intercept is $(0, -1/2)$

Asymptotes.

Vertical asymptotes occur where the denominator is zero and not cancelled:

$x + 2 = 0 \rightarrow x = -2$

$x - 1 = 0 \rightarrow x = 1$

So vertical asymptotes are:

$x = -2$ and $x = 1$

Horizontal asymptote.

Degree of numerator = 1

Degree of denominator = 2

Since degree of denominator is greater,

Horizontal asymptote is:

$y = 0$

12. (a) Find the area of the shaded region in the following figure in the form a/b.

From the figure, the shaded region is bounded by

the curve $y = x^2$,

the x axis,

the y axis,

and the vertical line $x = 4$.

Therefore, the required area is the area under the curve $y = x^2$ from $x = 0$ to $x = 4$.

Area

$= \int$ from 0 to 4 of $x^2\,dx$

$= [\, x^3 / 3 \,]_0^4$

$= 4^3 / 3 - 0$

$= 64 / 3$

Hence, the area of the shaded region is

64/3 square units.

12. (b) Verify that $\int$ from 0 to $\pi/3$ of $(\cos x - \sin x)\,dx = 1/2\,(\sqrt{3} - 1)$.

$\int (\cos x - \sin x)\,dx$

$= \int \cos x\,dx - \int \sin x\,dx$

$= \sin x + \cos x$

Evaluate from 0 to $\pi/3$.

At $x = \pi/3$:

$\sin \pi/3 + \cos \pi/3$

$= \sqrt{3}/2 + 1/2$

At $x = 0$:

$\sin 0 + \cos 0$

$= 0 + 1$

Therefore,

$\int_0^{\pi/3} (\cos x - \sin x)\, dx$

$= (\sqrt{3}/2 + 1/2) - 1$

$= (\sqrt{3} - 1) / 2$

Hence verified.

(c) Using the product rule, differentiate $y = x^2 \sin 2x$ with respect to x.

Let

$u = x^2$

$v = \sin 2x$

Then

$du/dx = 2x$

$dv/dx = 2 \cos 2x$

Using the product rule:

$dy/dx = u\, dv/dx + v\, du/dx$

$dy/dx = x^2(2 \cos 2x) + \sin 2x(2x)$

$\mathbf{dy/dx = 2x^2 \cos 2x + 2x \sin 2x}$

13. (a) A bag contains 4 white balls and 6 black balls. Two balls were selected from the bag, one after the other, without replacement. Draw a tree diagram showing all the probabilities and outcomes.

Total balls = 4 white (W) + 6 black (B) = 10

First draw:

P(W) = 4/10

P(B) = 6/10

Second draw if first is W:

Remaining balls = 3W and 6B, total = 9

P(W | W) = 3/9

P(B | W) = 6/9

Second draw if first is B:

Remaining balls = 4W and 5B, total = 9

P(W | B) = 4/9

P(B | B) = 5/9

Outcomes with probabilities:

WW = 4/10 × 3/9

WB = 4/10 × 6/9

BW = 6/10 × 4/9

BB = 6/10 × 5/9

(b) Using the tree diagram in part (a), find the probability that:

(i) the first ball is white and the second ball is black.

P(W then B)

= 4/10 × 6/9

= 24/90

= 4/15

(ii) one ball is white and the other is black.

This can occur as WB or BW.

P(WB or BW)

= P(WB) + P(BW)

= (4/10 × 6/9) + (6/10 × 4/9)

= 24/90 + 24/90

= 48/90

= 8/15

(c) Find the first four terms of the expansion of $(1 + x)^n$ in ascending powers of x.

Using the binomial expansion:

$(1 + x)^n$
$= 1 + nx + n(n - 1)/2\ x^2 + n(n - 1)(n - 2)/6\ x^3 + \ldots$

So the first four terms are:
$1 + nx + n(n - 1)/2\ x^2 + n(n - 1)(n - 2)/6\ x^3$

14. (a) The linear transformation

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & -1 \\ 1 & 3 & -2 \end{pmatrix}$$

takes the point A(x, y, z) to the point A′(3, 0, 15). Using the inverse matrix method, find the point

A(x, y, z).

We have
$T \cdot A = A'$

So,
$A = T^{-1} \cdot A'$

First find det(T).

det(T)
$= 1[\,(-2)(-2) - (-1)(3)\,]$
$- 1[\,2(-2) - (-1)(1)\,]$

- $1[\,2(3) - (-2)(1)\,]$

$= 1(4 + 3) - 1(-4 + 1) + 1(6 + 2)$
$= 7 + 3 + 8$
$= 18$

Adjoint of T (computed by cofactors):

$$adj(T) = \begin{pmatrix} 7 & 5 & 1 \\ 1 & -3 & -2 \\ -8 & -2 & -4 \end{pmatrix}$$

Therefore,

$T^{-1} = (1/18)\ adj(T)$

Now compute A:

$$A = (1/18)\ adj(T) = \begin{pmatrix} 7 & 5 & 1 \\ 1 & -3 & -2 \\ -8 & -2 & -4 \end{pmatrix} x \begin{pmatrix} 3 \\ 0 \\ 15 \end{pmatrix}$$

$x = (1/18)[7(3) + 5(0) + 1(15)]$

$= (1/18)(21 + 15)$

$= 36/18$

$= 2$

$y = (1/18)[1(3) - 3(0) - 2(15)]$

$= (1/18)(3 - 30)$

$= -27/18$

$= -3/2$

$z = (1/18)[-8(3) - 2(0) - 4(15)]$

$= (1/18)(-24 - 60)$

$= -84/18$

$= -14/3$

So,

$A(x, y, z) = (2, -3/2, -14/3)$

(b) The vertices of a triangle are P(1,1,1), Q(0,1,2), and R(3,2,1). Find the area of the triangle.

Find vectors:

PQ = Q − P

= (0 − 1, 1 − 1, 2 − 1)

= (−1, 0, 1)

PR = R − P

= (3 − 1, 2 − 1, 1 − 1)

= (2, 1, 0)

Cross product PQ × PR:

$$= \begin{pmatrix} i & j & k \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

= i(0 − 1) − j(0 − 2) + k(−1 − 0)

= −i + 2j − k

Magnitude:

|PQ × PR|

= √(1 + 4 + 1)

= √6

Area of triangle

= 1/2 × √6

(c) Given that

a = −3i + 7j + 5k,

b = −3i + 7j − 3k,

and c = 7i − 5j − 3k, find a · (b × c).

First find b × c.

b = (−3, 7, −3)

c = (7, −5, −3)

$$b \times c = \begin{pmatrix} i & j & k \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$= i[7(-3) - (-3)(-5)]$

$- j[(-3)(-3) - (-3)(7)]$

- $k[(-3)(-5) - 7(7)]$

$= i(-21 - 15) - j(9 + 21) + k(15 - 49)$

$= -36i - 30j - 34k$

Now find a · (b × c):

a · (b × c)

$= (-3, 7, 5) \cdot (-36, -30, -34)$

$= (-3)(-36) + 7(-30) + 5(-34)$

$= 108 - 210 - 170$

$= -272$

Therefore,

a · (b × c) = −272