THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL OF TANZANIA

CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

042 **ADDITIONAL MATHEMATICS**

Time: 3hour **SOLUTIONS** **Year: 2021**

Instructions

1. This paper consists of sections A and B with a total of fourteen (14) questions.
2. Answer all questions.
3. Section A carries sixty (60) marks and section B carries forty (40) marks.
4. All necessary working and answers for each question attempted must be shown clearly.
5. NECTA Mathematical tables and non-programmable calculators may be used.
6. All communication devices and any unauthorised materials are not allowed in the examination room.
7. Write your Examination Number on every page of your answer booklet(s).

1. (a) An object falls a vertical distance x which varies directly as the square of the time t. If it falls 900 cm in 20 seconds, write the variation equation expressing x in terms of t.

Since x varies directly as the square of t,

$x \propto t^2$

$x = kt^2$

Given that x = 900 when t = 20,

$900 = k \times 20^2$

$900 = k \times 400$

$k = 900 / 400$

$k = 9 / 4$

Therefore, the variation equation is:

$x = (9/4)t^2$

(b) It is given that y is inversely proportional to x^2. If y = 4 when x = 3, find the value of y when x is 6.

Since y is inversely proportional to x^2,

$y \propto 1 / x^2$

$y = k / x^2$

Given y = 4 when x = 3,

$4 = k / 9$

$k = 36$

When x = 6,

$y = 36 / 36$

$y = 1$

2. The following table shows yields of gold in tonnes produced by 100 traders at a certain mine in one day.

Gold in tonnes:	15–20	21–26	27–32	33–38	39–44	45–50
Frequency:	10	22	32	21	13	2

(a) Calculate the mean and mode if the assumed mean is 29.5 tonnes.

Class marks x:

17.5, 23.5, 29.5, 35.5, 41.5, 47.5

Let A = 29.5

Compute deviations $d = x - A$:

-12, -6, 0, 6, 12, 18

Compute fd:

10(-12) = -120

22(-6) = -132

32(0) = 0

21(6) = 126

13(12) = 156

2(18) = 36

Σfd = 66

Σf = 100

Mean = A + (Σfd / Σf)

Mean = 29.5 + 66/100

Mean = 30.16 tonnes

Modal class is 27–32 since it has the highest frequency 32.

Mode formula:

Mode = $l + [f_1 - f_0 / (2f_1 - f_0 - f_2)] \times h$

Where:

$l = 26.5$

$f_1 = 32$

$f_0 = 22$

$f_2 = 21$

$h = 6$

Mode = 26.5 + [32 − 22 / (64 − 22 − 21)] × 6

Mode = 26.5 + (10 / 21) × 6

Mode = 26.5 + 2.86

Mode = 29.36 tonnes

(b) Draw the cumulative frequency curve and from it calculate the semi-interquartile range.

Cumulative frequencies:

10, 32, 64, 85, 98, 100

Q1 position = 100 / 4 = 25

Q3 position = 3 × 100 / 4 = 75

Q1 lies in class 21–26

Q3 lies in class 33–38

$Q1 = 20.5 + [(25 - 10) / 22] \times 6$

$Q1 = 20.5 + (15/22) \times 6$

$Q1 = 24.59$

$Q3 = 32.5 + [(75 - 64) / 21] \times 6$

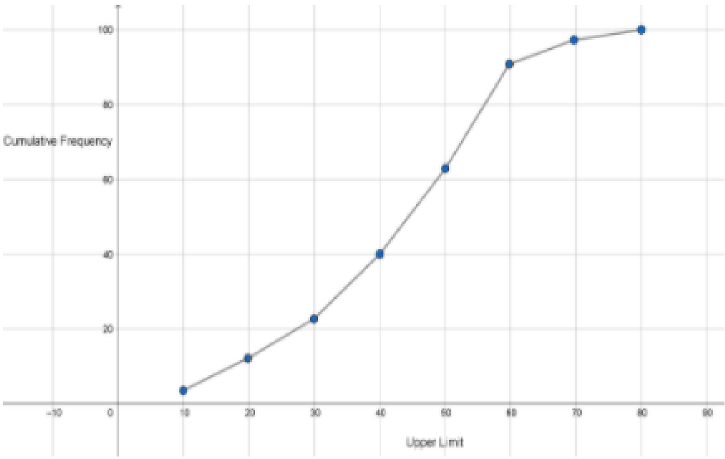
$Q3 = 32.5 + (11/21) \times 6$

$Q3 = 35.64$

Semi-interquartile range = $(Q3 - Q1) / 2$

$= (35.64 - 24.59) / 2$

= 5.53 tonnes



3. (a) If the points P(2,4), Q(3,y) and R(−3,4) are collinear, determine the value of y.

Slope of PR:

$(4 - 4) / (-3 - 2) = 0$

So line is horizontal, $y = 4$

Therefore, y = 4

(b) Determine the coordinates of the point dividing the line joining the point (2,3) and (4,6) in the ratio 1:3.

(i) Internally.

Using section formula:

$x = (1\times4 + 3\times2) / (1 + 3) = 10 / 4 = 2.5$

$y = (1\times6 + 3\times3) / (1 + 3) = 15 / 4 = 3.75$

Point = (2.5, 3.75)

(ii) Externally.

$x = (1\times4 - 3\times2) / (1 - 3) = -2 / -2 = 1$

$y = (1\times6 - 3\times3) / (1 - 3) = -3 / -2 = 1.5$

Point = (1, 1.5)

4. (a) Define "locus of a point" as applied in mathematics.

A locus of a point is the path traced by a moving point which moves according to a given rule or condition.

(b) The cartesian coordinates of the points A and B are (−3,0) and (3,0) respectively. If point P moves so that AP = 2PB, prove that its locus is the circle Z whose equation is $x^2 + y^2 - 10x + 9 = 0$.

Let P(x,y).

$AP^2 = (x + 3)^2 + y^2$

$PB^2 = (x - 3)^2 + y^2$

Given AP = 2PB,

$AP^2 = 4PB^2$

$(x + 3)^2 + y^2 = 4[(x - 3)^2 + y^2]$

Expand:

$x^2 + 6x + 9 + y^2 = 4x^2 - 24x + 36 + 4y^2$

Simplify:

$0 = 3x^2 - 30x + 3y^2 + 27$

Divide by 3:

$x^2 - 10x + y^2 + 9 = 0$

Hence the locus is the circle:

$\mathbf{x^2 + y^2 - 10x + 9 = 0}$

5. (a) Make t the subject of the formula A = (1 + t)^(1/2) / (1 − t).

$A(1 - t) = (1 + t)^{(1/2)}$

Square both sides:

$A^2(1 - t)^2 = 1 + t$

$A^2(1 - 2t + t^2) = 1 + t$

$A^2t^2 - 2A^2t + A^2 - 1 - t = 0$

$A^2t^2 - (2A^2 + 1)t + (A^2 - 1) = 0$

Solve for t using quadratic formula:

$\mathbf{t = [(2A^2 + 1) \pm \sqrt{((2A^2 + 1)^2 - 4A^2(A^2 - 1))}] / (2A^2)}$

(b) By using the substitution method, solve the following pair of simultaneous equations:

$x^2 + y = 18$

$y - 2x = -3$

From second equation:

$y = 2x - 3$

Substitute into first:

$x^2 + 2x - 3 = 18$

$x^2 + 2x - 21 = 0$

$(x + 7)(x - 3) = 0$

$x = 3$ or $x = -7$

If $x = 3$,

$y = 2(3) - 3 = 3$

If $x = -7$,

$y = 2(-7) - 3 = -17$

Solutions:

(3,3) and (−7, −17)

6. (a) A regular polygon is such that each interior angle is twice the exterior angle. What is the size of each interior angle and exterior angle?

   Let the exterior angle be x.

   Then the interior angle = 2x.

   For any polygon,

   interior angle + exterior angle = 180

   So,

   $2x + x = 180$

   $3x = 180$

   $x = 60$

   Exterior angle = 60 degrees.

   Interior angle = 2 × 60 = 120 degrees.

(b) (i) Indicate all lines of symmetry on a diagram of a regular pentagon by using dotted lines.

A regular pentagon has five equal sides and five equal angles. Each line of symmetry passes through one vertex and the midpoint of the opposite side. Therefore, five dotted lines should be drawn, each starting from a vertex and passing through the center to the midpoint of the opposite side.

(ii) State the order of rotational symmetry of the regular pentagon drawn in part (b)(i).

A regular pentagon fits onto itself five times during a full rotation of 360 degrees.

Order of rotational symmetry = 5.

7. (a) Show that $(\sin \theta + \sin 2\theta) / (1 + \cos \theta + \cos 2\theta) = \tan \theta$.

   Use trigonometric identities:

$\sin 2\theta = 2 \sin \theta \cos \theta$

$\cos 2\theta = 2 \cos^2\theta - 1$

Substitute into the expression:

Numerator:

$\sin \theta + \sin 2\theta$

$= \sin \theta + 2 \sin \theta \cos \theta$

$= \sin \theta(1 + 2 \cos \theta)$

Denominator:

$1 + \cos \theta + \cos 2\theta$

$= 1 + \cos \theta + (2 \cos^2\theta - 1)$

$= \cos \theta + 2 \cos^2\theta$

$= \cos \theta(1 + 2 \cos \theta)$

Therefore,

$(\sin \theta(1 + 2 \cos \theta)) / (\cos \theta(1 + 2 \cos \theta))$

Cancel $(1 + 2 \cos \theta)$:

$= \sin \theta / \cos \theta$

$= \tan \theta$

Hence proved.

(b) Derive the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$.

Consider a right angled triangle with hypotenuse 1.

Let the angle be θ.

Opposite side = $\sin \theta$

Adjacent side = $\cos \theta$

By Pythagoras theorem:

$(\text{opposite})^2 + (\text{adjacent})^2 = (\text{hypotenuse})^2$

$\sin^2\theta + \cos^2\theta = 1^2$

$\sin^2\theta + \cos^2\theta = 1$

Hence derived.

8. (a) (i) What is the rule governing the divisibility of any number by 9.

A number is divisible by 9 if and only if the sum of its digits is divisible by 9.

(ii) Show whether 1091524 is divisible by 9.

Sum of digits:

1 + 0 + 9 + 1 + 5 + 2 + 4

= 22

Since 22 is not divisible by 9,

1091524 is not divisible by 9.

(b) The following table shows the pattern of coefficients in the Pascal's triangle:

Power	Coefficients
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	?

How can the entry 20 in the sixth line be obtained? Write the entries in the seventh line.

In Pascal's triangle, each entry is obtained by adding the two numbers directly above it in the previous line.

The entry 20 in the sixth line is obtained as:

10 + 10 = 20

The seventh line is obtained by adding adjacent entries in the sixth line:

1

1 + 6 = 7

6 + 15 = 21

15 + 20 = 35

20 + 15 = 35

15 + 6 = 21

6 + 1 = 7

1

Therefore, the entries in the seventh line are:

1 7 21 35 35 21 7 1

9. (a) Prepare the truth table for

$[(\neg p \lor \neg q) \rightarrow \neg(p \land q)] \lor [(p \lor q) \rightarrow (\neg p \land \neg q)]$.

Truth table:

p	q	¬p	¬q	¬p∨¬q	p∧q	¬(p∧q)	(¬p∨¬q)→¬(p∧q)	p∨q	¬p∧¬q	(p∨q)→(¬p∧¬q)	S
T	T	F	F	F	T	F	T	T	F	F	T
T	F	F	T	T	F	T	T	T	F	F	T
F	T	T	F	T	F	T	T	T	F	F	T
F	F	T	T	T	F	T	T	F	T	T	T

The compound statement is a tautology.

(b) By using the laws of algebra of propositions, show whether the statement $p \land q$ logically implies $p \leftrightarrow q$.

$p \land q$ logically implies $p \leftrightarrow q$ if

$(p \land q) \rightarrow (p \leftrightarrow q)$ is a tautology.

$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$

So,

$(p \land q) \rightarrow [(p \rightarrow q) \land (q \rightarrow p)]$

If $p \land q$ is true, then p is true and q is true.

Thus, $p \rightarrow q$ is true and $q \rightarrow p$ is true.

Therefore, $(p \land q) \rightarrow (p \leftrightarrow q)$ is always true.

Hence, $p \land q$ logically implies $p \leftrightarrow q$.

10. (a) Given the universal set $\mu = \{1,2,3,\ldots,12\}$ and its subsets

$A = \{1,3,5,7\}$,

$B = \{2,3,4,5,6,8\}$,

$C = \{2,3,7,10\}$,

find the elements of $(A \cap B) \cup C$.

First find $A \cap B$:

$A \cap B = \{3,5\}$

Now find $(A \cap B) \cup C$:

$\{3,5\} \cup \{2,3,7,10\}$

$= \{2,3,5,7,10\}$

(b) If A, B and C are any three sets such that

$n(A) = 8$, $n(B) = 12$, $n(C) = 16$, $n(A \cap B) = 5$, $n(A \cap C) = 4$, $n(A \cup B \cup C) = 20$, and $n(A \cap B \cap C) = 2$, find $n(B \cap C)$.

Using the formula:

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C)\text{-}n(A \cap B \cap C)$

Substitute values:

$20 = 8 + 12 + 16$

$- 5 - 4 - n(B \cap C)$

$20 = 29 - n(B \cap C)$

$n(B \cap C) = 9$

11. (a) The roots of a quadratic equation $ax^2 + bx + c = 0$ are such that the first root is three times the second root. Show that $3b^2 = 16ac$.

Let the second root be α.

Then the first root $= 3\alpha$.

Sum of roots $= -b / a$

$\alpha + 3\alpha = 4\alpha = -b / a$

Product of roots $= c / a$

$\alpha \times 3\alpha = 3\alpha^2 = c / a$

From $4\alpha = -b / a$,

$\alpha = -b / (4a)$

Substitute into product:

$3(-b / 4a)^2 = c / a$

$3b^2 / 16a^2 = c / a$

Multiply both sides by $16a^2$:

$3b^2 = 16ac$

Hence shown that $3b^2 = 16ac$.

(b) When the function $f(x) = 2x^4 + kx^3 - 11x^2 + 4x + 12$ is divided by $x - 3$, the remainder is 60. Use the remainder theorem to compute the value of k.

By the remainder theorem,

$f(3) = 60$

Compute $f(3)$:

$f(3) = 2(3)^4 + k(3)^3 - 11(3)^2 + 4(3) + 12$

$= 2(81) + 27k - 11(9) + 12 + 12$

$= 162 + 27k - 99 + 24$

$= 87 + 27k$

Set equal to 60:

$87 + 27k = 60$

$27k = -27$

$k = -1$

(c) Sketch the graph of $f(x) = (x + 2) / (x^2 - 9)$.

Factor the denominator:

$x^2 - 9 = (x - 3)(x + 3)$

Vertical asymptotes:

$x = 3$ and $x = -3$

Horizontal asymptote:

Degree of numerator < degree of denominator,

so $y = 0$

Intercepts:

x-intercept:

$x + 2 = 0$

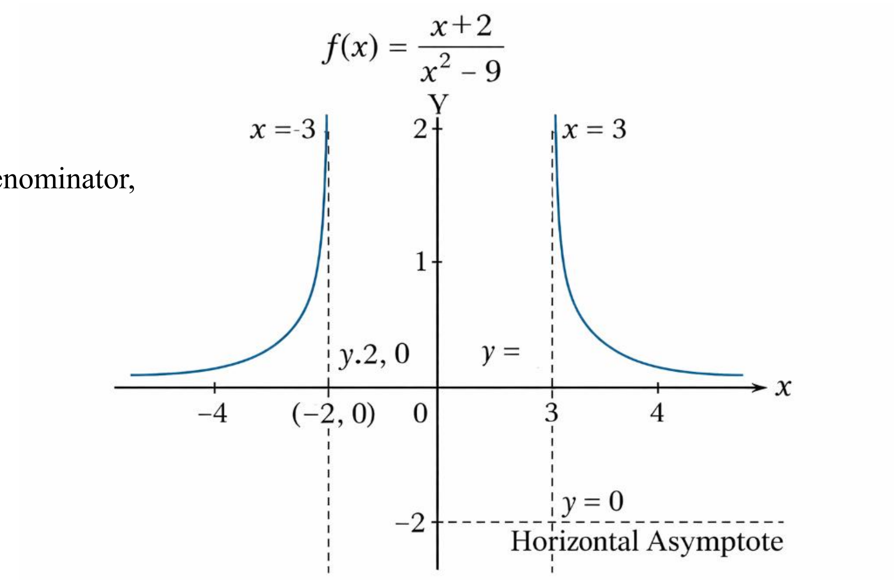
$x = -2$

y-intercept:

$x = 0$

$y = 2 / (-9)$

$y = -2/9$



The curve has two branches separated by the vertical asymptotes at $x = -3$ and $x = 3$, approaches the x-axis as $x \to \pm\infty$, crosses the x-axis at $(-2, 0)$, and passes through $(0, -2/9)$.

12. (a) Use the quotient rule to differentiate $((1 + x)^2) / (2 + x)$ with respect to x.

Let $y = (1 + x)^2 / (2 + x)$

Using the quotient rule:

$dy/dx = [(2(1 + x))(2 + x) - (1 + x)^2(1)] / (2 + x)^2$

Simplify numerator:

$2(1 + x)(2 + x) - (1 + x)^2$

$= (1 + x)[2(2 + x) - (1 + x)]$

$= (1 + x)(4 + 2x - 1 - x)$

$= (1 + x)(3 + x)$

Therefore,

$dy/dx = (1 + x)(x + 3) / (2 + x)^2$

(b) Given the curve $y = 2x^3 - 3x^2 - 36x + 3$:

(i) find the minimum value of y.

Differentiate:

$dy/dx = 6x^2 - 6x - 36$

Set $dy/dx = 0$:

$6x^2 - 6x - 36 = 0$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3$ or $x = -2$

Second derivative:

$d^2y/dx^2 = 12x - 6$

At $x = 3$:

$12(3) - 6 = 30 > 0$, minimum point

Find y at $x = 3$:

$y = 2(27) - 3(9) - 36(3) + 3$

$= 54 - 27 - 108 + 3$

$= -78$

Minimum value of $y = -78$.

(ii) determine the value of x at the point of inflexion.

Point of inflexion where second derivative = 0:

$12x - 6 = 0$

$x = 1/2$

(c) Compute the area enclosed by the curve $y = x^2 - 4$ and the x-axis.

Intercepts with x-axis:

$x^2 - 4 = 0$

$x = \pm 2$

Area = $\int$ from −2 to 2 of $(4 - x^2)$ dx

= $[4x - x^3/3]$ from −2 to 2

At x = 2:

8 − 8/3 = 16/3

At x = −2:

−8 + 8/3 = −16/3

Area = 16/3 − (−16/3)

= 32/3 square units

13. (a) A bag contains 8 green discs (G) and 4 blue discs (B). A disc is drawn and not replaced. A second disc is drawn. Copy and complete the following tree diagram then answer the questions that follow.

Total discs = 12

First draw:

P(G) = 8/12 = 2/3

P(B) = 4/12 = 1/3

Second draw:

If first is G:

Remaining G = 7, B = 4, total = 11

P(G | G) = 7/11

P(B | G) = 4/11

If first is B:

Remaining G = 8, B = 3, total = 11

P(G | B) = 8/11

P(B | B) = 3/11

Find the probability that:

(i) both discs are green.

P(GG) = 2/3 × 7/11

= 14/33

(ii) both discs are blue.

P(BB) = 1/3 × 3/11

= 1/11

(iii) one disc is green and one disc is blue.

P(GB or BG)

= (2/3 × 4/11) + (1/3 × 8/11)

= 8/33 + 8/33

= 16/33

(b) If A and B are dependent events whereby P(A) = 1/5, P(B) = 3/10 and P(A | B) = 1/10, find P(A ∪ B) and P(B ∩ A′).

P(A ∩ B) = P(A | B) × P(B)

= 1/10 × 3/10

= 3/100

P(A ∪ B) = P(A) + P(B) − P(A ∩ B)

= 1/5 + 3/10 − 3/100

= 20/100 + 30/100 − 3/100

= 47/100

P(B ∩ A′) = P(B) − P(A ∩ B)

= 3/10 − 3/100

= 27/100

(c) In how many ways can 11 people be seated on a bench if only 6 seats are available?

Number of ways = permutations of 11 people taken 6 at a time

= 11P6

= 11 × 10 × 9 × 8 × 7 × 6

= 332640

14. (a) Find the work done when a force given by F = 4i − 3j + 6k displaces an object from A(0,4,5) to B(3,12,10).

Displacement vector AB = B − A

AB = (3 − 0)i + (12 − 4)j + (10 − 5)k

AB = 3i + 8j + 5k

Work done W = F · AB

W = (4i − 3j + 6k) · (3i + 8j + 5k)

$W = 4(3) + (-3)(8) + 6(5)$

$W = 12 - 24 + 30$

$W = 18$

Work done = 18 units.

(b) The position vectors of the points A and B are $a = 5i - j - 3k$ and $b = i + 3j - 5k$ respectively. Show that vector $a + b$ is perpendicular to vector $a - b$.

First find $a + b$:

$a + b = (5 + 1)i + (-1 + 3)j + (-3 - 5)k$

$a + b = 6i + 2j - 8k$

Now find $a - b$:

$a - b = (5 - 1)i + (-1 - 3)j + (-3 + 5)k$

$a - b = 4i - 4j + 2k$

Find the dot product:

$(a + b) \cdot (a - b)$

$= (6)(4) + (2)(-4) + (-8)(2)$

$= 24 - 8 - 16$

$= 0$

Since the dot product is zero,

vector a + b is perpendicular to vector a − b.

(c) Determine the image of $(3, -8)$ under a reflection in the line $x + y = 0$ followed by a rotation of -90 degrees clockwise about the origin.

The line $x + y = 0$ is $y = -x$.

Reflection in $y = -x$ maps (x, y) to $(-y, -x)$.

Reflect $(3, -8)$:

$(x, y) \rightarrow (-(-8), -3)$

$= (8, -3)$

Now rotate $(8, -3)$ by -90 degrees clockwise about the origin.

A rotation of -90 degrees maps (x, y) to $(y, -x)$.

So,

$(8, -3) \rightarrow (-3, -8)$

Final image $= (-3, -8)$.