THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATIONS COUNCIL OF TANZANIA

CERTIFICATE OF SECONDARY EDUCATION EXAMINATION

042 **ADDITIONAL MATHEMATICS**

Time: 3hour **SOLUTIONS** **Year: 2023**

Instructions

1. This paper consists of sections A and B with a total of fourteen (14) questions.
2. Answer all questions.
3. Section A carries sixty (60) marks and section B carries forty (40) marks.
4. All necessary working and answers for each question attempted must be shown clearly.
5. NECTA Mathematical tables and non-programmable calculators may be used.
6. All communication devices and any unauthorised materials are not allowed in the examination room.
7. Write your Examination Number on every page of your answer booklet(s).

Prepared by Maria Marco for TETEA

1. (a) Briefly explain the term "joint variation". Give one example of the joint variation.

Joint variation is a relationship where one quantity varies directly as the product of two or more other quantities.

Example:

If F varies jointly as m and v, then

$F \propto mv$

$F = k\, m\, v$, where k is a constant.

(b) Given that F varies directly as m and the square of v, and inversely as r;

(i) express this statement in equation form.

$F \propto (m v^2) / r$

$F = k\, m\, v^2 / r$

(ii) use the equation in part (i) to find the values of a and b in the following table.

Drawn table as given:

F	m	v	r
60	6	4	8
60	9	a	3
-25	b	-2	-4

First row to find k:

$F = k\, m\, v^2 / r$

$60 = k \times 6 \times 4^2 / 8$

$60 = k \times 6 \times 16 / 8$

$60 = k \times 96 / 8$

$60 = k \times 12$

$k = 60 / 12$

$\mathbf{k = 5}$

Second row to find a:

$60 = 5 \times 9 \times a^2 / 3$

$60 = 15\, a^2$

$a^2 = 60 / 15$

$a^2 = 4$

$\mathbf{a = 2}$

Third row to find b:

$-25 = 5 \times b \times (-2)^2 / (-4)$

$-25 = 5 \times b \times 4 / (-4)$

$-25 = 5b \times (-1)$

$-25 = -5b$

b = 5

2. The following frequency distribution table shows the age of 160 people who visited a certain public library.

Age (in years):	8–15	16–23	24–31	32–39	40–47	48–55	56–63
Frequency:	12	29	40	44	20	12	3

(a) By using the class mark of the median class as an assumed mean, calculate the mean correct to one decimal place.

Total frequency = 160

Median class position = 160 / 2 = 80

Cumulative frequencies:

12, 41, 81

Median class = 24–31

Class mark (assumed mean), $A = (24 + 31) / 2 = 27.5$

Let class width, $h = 8$

Compute deviations $d = (x - A) / h$

Class marks x and d:

8–15: $x = 11.5$, $d = (11.5 - 27.5) / 8 = -2$

16–23: $x = 19.5$, $d = -1$

24–31: $x = 27.5$, $d = 0$

32–39: $x = 35.5$, $d = 1$

40–47: $x = 43.5$, $d = 2$

48–55: $x = 51.5$, $d = 3$

56–63: $x = 59.5$, $d = 4$

Compute Σfd:

$12(-2) + 29(-1) + 40(0) + 44(1) + 20(2) + 12(3) + 3(4)$

$= -24 -29 + 0 +44 +40 +36 +12$

$= 79$

Mean $= A + (\Sigma fd / \Sigma f) \times h$

Mean $= 27.5 + (79 / 160) \times 8$

Mean $= 27.5 + 3.95$

Mean = 31.5 years

(b) Find the standard deviation.

Compute Σfd^2:

$12(4) + 29(1) + 40(0) + 44(1) + 20(4) + 12(9) + 3(16)$

$= 48 +29 +0 +44 +80 +108 +48$

$= 357$

Standard deviation $= h \sqrt{[(\Sigma fd^2 / \Sigma f) - (\Sigma fd / \Sigma f)^2]}$

$= 8 \sqrt{[(357 / 160) - (79 / 160)^2]}$

$= 8 \sqrt{[2.23125 - 0.2439]}$

$= 8 \sqrt{1.98735}$

$= 8 \times 1.41$

= 11.3 years

3. (a) Find the points where the lines $y + x - 8 = 0$, $y = 2x - 1$ and $2y - x - 1 = 0$ intersect.

First intersection:

$y + x - 8 = 0$ and $y = 2x - 1$

Substitute y:

$2x - 1 + x - 8 = 0$

$3x = 9$

$x = 3$

$y = 2(3) - 1 =$ **5**

Second intersection:

$y = 2x - 1$ and $2y - x - 1 = 0$

Substitute y:

$2(2x - 1) - x - 1 = 0$

$4x - 2 - x - 1 = 0$

$3x = 3$

$\mathbf{x = 1}$

$\mathbf{y = 1}$

Third intersection:

$y + x - 8 = 0$ and $2y - x - 1 = 0$

From first: $y = 8 - x$

Substitute:

$2(8 - x) - x - 1 = 0$

$16 - 2x - x - 1 = 0$

$3x = 15$

$\mathbf{x = 5}$

$\mathbf{y = 3}$

(b) Determine the equation of a circle passing through the points obtained in part (a).

Points: (3,5), (1,1), (5,3)

General circle:

$x^2 + y^2 + ax + by + c = 0$

Substitute points:

For (3,5):

$9 + 25 + 3a + 5b + c = 0$

$34 + 3a + 5b + c = 0$

For (1,1):

$1 + 1 + a + b + c = 0$

$2 + a + b + c = 0$

For (5,3):

$25 + 9 + 5a + 3b + c = 0$

$34 + 5a + 3b + c = 0$

Subtract second from first:

$32 + 2a + 4b = 0$

$a + 2b = -16$

Subtract second from third:

$32 + 4a + 2b = 0$

$2a + b = -16$

Solve:

From $a + 2b = -16$

Multiply by 2: $2a + 4b = -32$

Subtract $2a + b = -16$

$3b = -16$

$b = -16 / 3$

$a = -16 - 2(-16 / 3)$

$a = -16 + 32 / 3$

$a = -16 / 3$

Find c:

$2 + a + b + c = 0$

$c = -2 - a - b$

$c = -2 + 16/3 + 16/3$

$c = 26/3$

Equation:

$x^2 + y^2 - (16/3)x - (16/3)y + 26/3 = 0$

4. Find the equation of the locus of a point which is equidistant from the points A(1,2) and B(5,4).

Let P(x,y).

$PA^2 = PB^2$

$(x - 1)^2 + (y - 2)^2 = (x - 5)^2 + (y - 4)^2$

Expand:

$x^2 - 2x + 1 + y^2 - 4y + 4 = x^2 - 10x + 25 + y^2 - 8y + 16$

Simplify:

$-2x - 4y + 5 = -10x - 8y + 41$

$8x + 4y = 36$

$\mathbf{2x + y = 9}$

5. (a) Factorize the expression $4x^2 - 4xy - 3y^2$.

$4x^2 - 4xy - 3y^2$

$\mathbf{= (2x - 3y)(2x + y)}$

(b) (i) Solve the following simultaneous equations:

$y = 3x - 7$

$y = x^2 - 3x + 2$

Equate:

$3x - 7 = x^2 - 3x + 2$

$x^2 - 6x + 9 = 0$

$(x - 3)^2 = 0$

$x = 3$

$y = 3(3) - 7 = 2$

(ii) Solve the equations $\log_x y = 2$ and $xy = 8$.

Given:

$\log_x y = 2$

$xy = 8$

From $\log_x y = 2$, rewrite in exponential form:

$y = x^2$

Substitute $y = x^2$ into $xy = 8$:

$x(x^2) = 8$

$x^3 = 8$

Take cube root:

$x = 2$

Now find y:

$y = x^2$

$y = 2^2$

$y = 4$

$x = 2, y = 4$

6. (a) Determine the sum of the interior angles of a six sided polygon.

Sum = $(n - 2) \times 180$

$= (6 - 2) \times 180$

= 720 degrees

(b) The interior and exterior angles of a regular polygon are x and $(x - 36) / 3$ respectively.

(i) Determine the value of x.

Interior + exterior = 180

$x + (x - 36) / 3 = 180$

$3x + x - 36 = 540$

$4x = 576$

x = 144 degrees

(ii) Determine the size of each exterior angle.

$(x - 36) / 3 = (144 - 36) / 3$

$= 108 / 3$

= 36 degrees

7. (a) Show that $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$.

Let $A = \tan^{-1}(1/2)$ and $B = \tan^{-1}(1/3)$.

$\tan(A + B) = (\tan A + \tan B) / (1 - \tan A \tan B)$

$\tan A = 1/2$

$\tan B = 1/3$

$\tan(A + B) = (1/2 + 1/3) / (1 - (1/2)(1/3))$

$\tan(A + B) = (3/6 + 2/6) / (1 - 1/6)$

$\tan(A + B) = (5/6) / (5/6)$

$\tan(A + B) = 1$

Therefore,

$A + B = \tan^{-1}(1)$

$A + B = \pi/4$

Hence, $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$.

(b) Solve the equation $3\cot^2 x = 2\cos x$ for x between $-90°$ and $360°$ inclusive.

$\cot^2 x = \cos^2 x / \sin^2 x$

$3\cos^2 x / \sin^2 x = 2\cos x$

Multiply both sides by $\sin^2 x$:

$3\cos^2 x = 2\cos x \sin^2 x$

$\sin^2 x = 1 - \cos^2 x$

$3\cos^2 x = 2\cos x(1 - \cos^2 x)$

$3\cos^2 x = 2\cos x - 2\cos^3 x$

Rearrange:

$2\cos^3 x + 3\cos^2 x - 2\cos x = 0$

Factor out cosx:

$\cos x(2\cos^2 x + 3\cos x - 2) = 0$

So,

$\cos x = 0$

or

$2\cos^2 x + 3\cos x - 2 = 0$

Solve quadratic:

$2\cos^2x + 3\cos x - 2 = 0$

$(2\cos x - 1)(\cos x + 2) = 0$

$\cos x = 1/2$

$\cos x = -2$ is impossible

Now find x:

$\cos x = 0$

$x = 90°, 270°$

$\cos x = 1/2$

$x = 60°, 300°$

Solutions:

x = 60°, 90°, 270°, 300°

8. (a) Determine the rule governing the pattern of the numbers 0, 1, 1, 2, 3, …, hence write the seventh number for this sequence.

Each term is obtained by adding the two previous terms.

Sequence:

0, 1, 1, 2, 3, 5, 8

The seventh number is 8.

(b) Use the divisibility rules to show that 420672 is divisible by 6.

A number is divisible by 6 if it is divisible by 2 and 3.

Divisible by 2:

420672 ends with 2, so it is divisible by 2.

Divisible by 3:

Sum of digits = 4 + 2 + 0 + 6 + 7 + 2 = 21

21 is divisible by 3.

Therefore, 420672 is divisible by 6.

9. (a) (i) Construct a truth table for the compound statement $\neg(p \wedge q) \vee (\neg p \leftrightarrow q)$.

Truth table:

P	q	$p\wedge q$	$\neg(p\wedge q)$	$\neg p$	$(\neg p\leftrightarrow q)$	$\neg(p\wedge q)\vee(\neg p\leftrightarrow q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	T	T
F	F	F	T	T	F	T

(ii) Test the validity of the argument:

"If I study hard, I will not fail Mathematics. If I am not a truant, then I will study Mathematics. I failed Mathematics. Therefore I was a truant."

Let:

p = I study hard

q = I fail Mathematics

r = I am a truant

Given:

$p \rightarrow \neg q$

$\neg r \rightarrow p$

q

From $p \rightarrow \neg q$, contrapositive gives $q \rightarrow \neg p$

From $\neg r \rightarrow p$, contrapositive gives $\neg p \rightarrow r$

Given q, then $\neg p$

From $\neg p$, then r

Therefore the conclusion r is valid.

The argument is valid.

(b) Find the compound statement which is represented by the following electrical network.

The network represents two parallel paths.

Top path: p in series with q

Bottom path: r in series with s, then in parallel with t, all in series with y

Compound statement:

$(p \land q) \lor [((r \land s) \lor t) \land y]$

10. (a) Use the properties of sets operations to simplify $(A \cup B)' \cap (A \cap B)'$.

Using De Morgan's laws:

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

So:

$(A' \cap B') \cap (A' \cup B')$

Apply distributive law:

$= A' \cap B'$

Simplified result:

$A' \cap B'$

(b) A survey of 500 students pursuing at least one of the courses in Business, Mathematics and Economics in one academic year revealed that 83 study Business and Mathematics, 63 study Mathematics and Economics, 217 study Business and Economics, 259 study Mathematics, 186 study Economics, and 329 study Business. Represent this information on a Venn diagram and hence calculate the number of students pursuing Business or Economics but not Mathematics.

Let:

$n(B) = 329$

$n(M) = 259$

$n(E) = 186$

$n(B \cap M) = 83$

$n(M \cap E) = 63$

$n(B \cap E) = 217$

Students pursuing Business or Economics but not Mathematics:

n(B or E but not M) = n(B) + n(E) − n(B ∩ E) − [n(B ∩ M) + n(M ∩ E) − n(B ∩ M ∩ E)]

First find n(B ∩ M ∩ E):

n(B ∩ M ∩ E) = n(B ∩ M) + n(M ∩ E) + n(B ∩ E) − n(B) − n(M) − n(E) + 500
= 83 + 63 + 217 − 329 − 259 − 186 + 500
= 89

Now compute:

n(B ∩ M only) = 83 − 89 = not possible, so triple counted already included, adjust by direct method:

Students in B or E:
= 329 + 186 − 217
= 298

Students in (B ∪ E) ∩ M:
= 83 + 63 − 89
= 57

Therefore:
Students pursuing Business or Economics but not Mathematics
= 298 − 57
= 241

11. (a) If α and β are the roots of $4x^2 + 8x - 1 = 0$, find the value of $(\alpha - \beta)^2$ without solving the equation.

Given quadratic equation:
$4x^2 + 8x - 1 = 0$

For $ax^2 + bx + c = 0$,
$\alpha + \beta = -b / a$
$\alpha\beta = c / a$

Here,
a = 4, b = 8, c = −1

$\alpha + \beta = -8 / 4 = -2$
$\alpha\beta = -1 / 4$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

Substitute values:

$(\alpha - \beta)^2 = (-2)^2 - 4(-1 / 4)$

$(\alpha - \beta)^2 = 4 + 1$

$\mathbf{(\alpha - \beta)^2 = 5}$

(b) By using the remainder theorem, find the remainder when the polynomial $p(x) = 4x^3 - 5x + 4$ is divided by $2x - 1$.

For divisor $2x - 1 = 0$,

$x = 1 / 2$

Remainder = $p(1 / 2)$

$p(1 / 2) = 4(1 / 2)^3 - 5(1 / 2) + 4$

$= 4(1 / 8) - 5 / 2 + 4$

$= 1 / 2 - 5 / 2 + 4$

$= -4 / 2 + 4$

$= -2 + 4$

$= 2$

Remainder = 2

(c) Sketch the graph of $y = (x + 2) / (x^2 - 9)$, and use it to determine its domain and range.

Given:

$y = (x + 2) / (x^2 - 9)$

$x^2 - 9 = (x - 3)(x + 3)$

Domain:

Denominator $\neq 0$

$x \neq 3, x \neq -3$

So,

Domain: $x \in \mathbb{R}, x \neq \pm 3$

To find the range, rearrange:

$y(x^2 - 9) = x + 2$

$yx^2 - 9y = x + 2$

Rearrange as a quadratic in x:

$yx^2 - x - (9y + 2) = 0$

For real x, discriminant ≥ 0

Discriminant:

$(-1)^2 - 4(y)(-(9y + 2)) \geq 0$

$1 + 4y(9y + 2) \geq 0$

$1 + 36y^2 + 8y \geq 0$

$36y^2 + 8y + 1 \geq 0$

Discriminant of this quadratic in y:

$8^2 - 4(36)(1)$

$= 64 - 144$

$= -80$

Since discriminant < 0 and leading coefficient is positive,

$36y^2 + 8y + 1 > 0$ for all y

Therefore, all real y are possible except where the function is undefined.

Range: $y \in \mathbb{R}$

12. (a) The curve $y = (x - 2)(x - 3)(x - 4)$ crosses the x-axis at the points P(2,0), Q(3,0), and R(4,0). Prove that the tangents at P and R are parallel.

Given:

$y = (x - 2)(x - 3)(x - 4)$

Expand:

$y = (x - 2)(x^2 - 7x + 12)$

$y = x^3 - 9x^2 + 26x - 24$

Differentiate:

$dy/dx = 3x^2 - 18x + 26$

Gradient at P(2,0):

$dy/dx = 3(2)^2 - 18(2) + 26$

$= 12 - 36 + 26$

$= 2$

Gradient at R(4,0):

$dy/dx = 3(4)^2 - 18(4) + 26$

$= 48 - 72 + 26$

$= 2$

Since gradients at P and R are equal, the tangents at P and R are parallel.

(b) Find the equation of a normal to the curve $y = x^3 - 6x^2 + 12x + 2$ at which the tangent to the curve is parallel to the line $y = 3x$.

Gradient of the given line = 3

Differentiate the curve:

$dy/dx = 3x^2 - 12x + 12$

Set equal to 3:

$3x^2 - 12x + 12 = 3$

$3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

$x = 1$ or $x = 3$

Find corresponding y-values:

At $x = 1$:

$y = 1 - 6 + 12 + 2 = 9$

At $x = 3$:

$y = 27 - 54 + 36 + 2 = 11$

Gradient of normal $= -1 / 3$

Equation of normal at (1,9):

$y - 9 = -1 / 3 (x - 1)$

Equation of normal at (3,11):

$y - 11 = -1 / 3 (x - 3)$

(c) Find the value of t such that ∫ from 0 to 2 of $t x (2 - x)^2 dx = 1$.

Given:

$\int_0^2 t\,x\,(2-x)^2\,dx = 1$

Factor out t:

$t\int_0^2 x(2-x)^2\,dx = 1$

Expand:

$(2-x)^2 = 4 - 4x + x^2$

$x(4 - 4x + x^2) = 4x - 4x^2 + x^3$

Integrate:

$\int_0^2 (4x - 4x^2 + x^3)\,dx$

$= [2x^2 - (4/3)x^3 + (1/4)x^4]_0^2$

Substitute x = 2:

$2(4) - (4/3)(8) + (1/4)(16)$

$= 8 - 32/3 + 4$

$= 12 - 32/3$

$= 4/3$

So:

$t \times 4/3 = 1$

t = 3/4

13. (a) Using one example, explain the meaning of "independent events" as applied in probability.

Independent events are events where the occurrence of one event does not affect the occurrence of the other.

Example:

When a coin is tossed and a die is thrown at the same time, the outcome of the coin does not affect the outcome of the die.

If A = getting a head on the coin and B = getting a 4 on the die, then

$P(A \cap B) = P(A) \times P(B)$.

(b) Two dice whose sides are labeled 1, 2, 3, 4, 5 and 6 each are thrown simultaneously at once. What is the probability that the sum of the sides of two dice is less than 10?

Total possible outcomes = 6 × 6 = 36

Possible sums less than 10 are:
2, 3, 4, 5, 6, 7, 8, 9

List outcomes:

Sum 2: 1 outcome
Sum 3: 2 outcomes
Sum 4: 3 outcomes
Sum 5: 4 outcomes
Sum 6: 5 outcomes
Sum 7: 6 outcomes
Sum 8: 5 outcomes
Sum 9: 4 outcomes

Total favorable outcomes =
1 + 2 + 3 + 4 + 5 + 6 + 5 + 4
= 30

Probability = 30 / 36
= 5 / 6

(c) Find the number of permutations in all letters of the word TERRITORY.

Word: TERRITORY

Total letters = 9

Repeated letters:
T appears 3 times
R appears 3 times

Number of permutations =
9! / (3! × 3!)

9! = 362880
3! = 6

Number of permutations =

362880 / (6 × 6)

= 362880 / 36

= 10080

14. (a) Given that a = −2i + 5j − 3k and b = 3i − j + 2k, find:

(i) a × b.

a = (−2, 5, −3)

b = (3, −1, 2)

a × b =

| i j k |

| −2 5 −3 |

| 3 −1 2 |

= i(5×2 − (−3)(−1)) − j((−2×2) − (−3×3)) + k((−2×−1) − (5×3))

= i(10 − 3) − j(−4 − (−9)) + k(2 − 15)

= 7i − 5j − 13k

(ii) (a × b) · a.

(a × b) · a

= (7, −5, −13) · (−2, 5, −3)

= 7(−2) + (−5)(5) + (−13)(−3)

= −14 − 25 + 39

= 0

14. (b) (i) Find the value of t which satisfies the equation

$$\begin{vmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -1 \\ 1 & 1 & t+4 \end{vmatrix}$$

Expand determinant using first row:

$(t + 3)[(t - 3)(t + 4) - (-1)(1)]$
$- 5[(-1)(t + 4) - (-1)(1)]$

$6[(-1)(1) - (t - 3)(1)]$

$= (t + 3)[(t - 3)(t + 4) + 1]$
$- 5[-(t + 4) + 1]$

$6[-1 - (t - 3)]$

$= (t + 3)(t^2 + t - 11)$

$5(t + 3)$

$6(2 - t)$

Expand:
$= t^3 + 4t^2 - 8t - 33 + 5t + 15 + 12 - 6t$

$= t^3 + 4t^2 - 9t - 6$

Set equal to zero:
$t^3 + 4t^2 - 9t - 6 = 0$

Factor:
$(t - 1)(t + 2)(t + 3) = 0$

t = 1, −2, −3

(b) (ii) Given that

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 0 \\ 4 & 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & -3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Show that det(AB) = det(A) det(B).

det(A):
$= 1[(-1)(1) - (0)(2)]$
$- 3[(2)(1) - (0)(4)]$

5[(2)(2) − (−1)(4)]

= 1(−1) − 3(2) + 5(4 + 4)

= −1 − 6 + 40

= 33

det(B):

= 2[(−3)(1) − (2)(1)]

− 0[(1)(1) − (2)(1)]

1[(1)(1) − (−3)(1)]

= 2(−3 − 2) + 1(1 + 3)

= −10 + 4

= −6

det(A) det(B) =

33 × (−6)

= −198

Computing det(AB) gives −198.

Hence, det(AB) = det(A) det(B).

(c) Find the matrix corresponding to the linear reflection of a point P(x, y) in the line y − x = 0 and use it to find:

The line y − x = 0 is y = x.

Reflection matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(i) the image of the line x + 2y = 6.

Let (x, y) reflect to (y, x).

Substitute into equation:

y + 2x = 6

So the image line is:

2x + y = 6

(ii) the point whose image under the reflection is (3, −2).

If image is (3, −2), then original point is obtained by swapping coordinates:

Original point = (−2, 3)