THE UNITED REPUBLIC OF TANZANIA

NATIONAL EXAMINATION COUNCIL

DIPLOMA IN SECONDARY EDUCATION EXAMINATION

740 **MATHEMATICS**

Time: 3 Hours. **ANSWER** **Year: 2005**

Instructions

1. This paper consists of sections A, B and C.
2. Answer **all** questions from Section A and **two (2)** questions from each of section B and C.
3. Section A and B carry **30** marks, Section C carry 40 marks.
4. Mathematical tables and non-programmable calculators may be used
4. Cellular phones are **not** allowed inside the examination room.
5. Write your **Examination Number** on every page of your answer booklet

SECTION A (49 marks)

Answer all questions in this section.

1.(a) Find the differential coefficient of

$y = \sec^{-1}((x + 1)/(x - 1)) + \sin^{-1}((x - 1)/(x + 1))$.

Let $y_1 = \sec^{-1}((x + 1)/(x - 1))$

Let $y_2 = \sin^{-1}((x - 1)/(x + 1))$

Differentiate y_1 with respect to x

$d/dx[\sec^{-1} u] = u' / (|u|\sqrt{(u^2 - 1)})$

$u = (x + 1)/(x - 1)$

$u' = [(x - 1)(1) - (x + 1)(1)] / (x - 1)^2$

$u' = (x - 1 - x - 1) / (x - 1)^2$

$u' = -2 / (x - 1)^2$

$u^2 - 1 = ((x + 1)^2 - (x - 1)^2) / (x - 1)^2$

$u^2 - 1 = (x^2 + 2x + 1 - x^2 + 2x - 1) / (x - 1)^2$

$u^2 - 1 = 4x / (x - 1)^2$

$\sqrt{(u^2 - 1)} = 2\sqrt{x} / |x - 1|$

$dy_1/dx = [-2 / (x - 1)^2] / [((x + 1)/(x - 1)) \times (2\sqrt{x} / |x - 1|)]$

$dy_1/dx = -1 / (\sqrt{x}\,(x + 1))$

Differentiate y_2 with respect to x

$d/dx[\sin^{-1} v] = v' / \sqrt{(1 - v^2)}$

$v = (x - 1)/(x + 1)$

$v' = [(x + 1)(1) - (x - 1)(1)] / (x + 1)^2$

$v' = (x + 1 - x + 1) / (x + 1)^2$

$v' = 2 / (x + 1)^2$

$1 - v^2 = ((x + 1)^2 - (x - 1)^2) / (x + 1)^2$

$1 - v^2 = 4x / (x + 1)^2$

$\sqrt{(1 - v^2)} = 2\sqrt{x} / |x + 1|$

$dy_2/dx = [2 / (x + 1)^2] / [2\sqrt{x} / |x + 1|]$

$dy_2/dx = 1 / (\sqrt{x}\,(x + 1))$

Therefore

$dy/dx = dy_1/dx + dy_2/dx$

$dy/dx = -1 / (\sqrt{x}\,(x + 1)) + 1 / (\sqrt{x}\,(x + 1))$

$dy/dx = 0$

2. Solve for x if $\sqrt{(2x - 1)} - \sqrt{(x - 1)} = 1$.

$\sqrt{(2x - 1)} = 1 + \sqrt{(x - 1)}$

Square both sides

$2x - 1 = 1 + 2\sqrt{(x - 1)} + (x - 1)$

$2x - 1 = x + 2\sqrt{(x - 1)}$

$x - 1 = 2\sqrt{(x - 1)}$

Square again

$(x - 1)^2 = 4(x - 1)$

$x^2 - 2x + 1 = 4x - 4$

$x^2 - 6x + 5 = 0$

$(x - 5)(x - 1) = 0$

$x = 5$ or $x = 1$

Check in original equation

At $x = 1$
$\sqrt{(1)} - \sqrt{(0)} = 1$ ✓

At $x = 5$
$\sqrt{(9)} - \sqrt{(4)} = 3 - 2 = 1$ ✓

Therefore x = 1 or 5.

3. The angle of a sector AOB shown in the diagram below is θ radians and the radius of the circle is r cm. The area of the sector is 10 cm^2 and its total perimeter is 13 cm. Find r and θ.

Area of sector
$= 1/2\ r^2\theta$

$1/2 \, r^2\theta = 10$

$r^2\theta = 20$

Perimeter of sector

$= 2r + r\theta$

$2r + r\theta = 13$

From $r^2\theta = 20$

$\theta = 20 / r^2$

Substitute into perimeter equation

$2r + r(20 / r^2) = 13$

$2r + 20/r = 13$

Multiply by r

$2r^2 + 20 = 13r$

$2r^2 - 13r + 20 = 0$

$(2r - 5)(r - 4) = 0$

$r = 4$ or $r = 5/2$

Take $r = 4$

$\theta = 20 / 16$

$\theta = 5/4$ radians

4. Simplify as far as possible

$[(a^2r^{-3}\ b^{-1}\ s\ a^{-5}\ r)^t\ (ab^{-1})^t] / (a^{-r+2}t\ b^2t)^{-1}$

Combine powers in the numerator

$a^2 \times a^{-5} \times a^1 \times a^1 = a^{-1}$

$r^{-3} \times r^1 = r^{-2}$

$b^{-1} \times b^{-1} = b^{-2}$

Numerator becomes

$(a^{-1}\ r^{-2}\ b^{-2})^t$

Denominator

$(a^{-r+2}t\ b^2t)^{-1} = a^{r-2}t\ b^{-2}t$

Final result

$a^{-1}t^{+r-2}t\ r^{-2}t\ b^{-2}t^{-2}$

5.(a) $\int \cos^2x \sin^3x\, dx$.

$\sin^3x = \sin^2x \sin x$

$\sin^2x = 1 - \cos^2x$

$\int \cos^2x\,(1 - \cos^2x) \sin x\, dx$

Let $u = \cos x$

$du = -\sin x\, dx$

Integral becomes

$-\int u^2(1 - u^2)\, du$

$-\int (u^2 - u^4)\, du$

$-(u^3/3 - u^5/5) + C$

$= -u^3/3 + u^5/5 + C$

Substitute $u = \cos x$

$\mathbf{= -\cos^3 x/3 + \cos^5 x/5 + C}$

(b) Express e^i in the form $x + iy$.

$e^i = \cos 1 + i \sin 1$

$x = \cos 1$
$y = \sin 1$

6.(a) Solve the following linear systems by using determinants

$-2x_1 + 3x_2 - x_3 = 1$
$x_1 + 2x_2 - x_3 = 4$
$-2x_1 - x_2 + x_3 = -3$

Coefficient matrix determinant

| −2 3 −1 |
| 1 2 −1 |
| −2 −1 1 |

$\Delta = -2(2\cdot1 - (-1)(-1)) - 3(1\cdot1 - (-1)(-2)) - 1(1\cdot(-1) - 2(-2))$

$\Delta = -2(2 - 1) - 3(1 - 2) - (-1 + 4)$

$\Delta = -2 + 3 - 3$

$\Delta = -2$

$\Delta_1 = | 1\ 3\ {-1} |$

$| 4\ 2\ {-1} |$

$| {-3}\ {-1}\ 1 |$

$\Delta_1 = 1(2\cdot1 - (-1)(-1)) - 3(4\cdot1 - (-1)(-3)) - 1(4(-1) - 2(-3))$

$\Delta_1 = 1(2 - 1) - 3(4 - 3) - (-4 + 6)$

$\Delta_1 = 1 - 3 - 2$

$\Delta_1 = -4$

$x_1 = \Delta_1 / \Delta = 2$

Similarly

$x_2 = 1$

$x_3 = 3$

7.(a) Show that $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$.

Let $\alpha = \tan^{-1}(1/2)$

Let $\beta = \tan^{-1}(1/3)$

$\tan(\alpha + \beta) = (1/2 + 1/3) / (1 - 1/6)$

$\tan(\alpha + \beta) = (5/6) / (5/6)$

$\tan(\alpha + \beta) = 1$

$\alpha + \beta = \pi/4$

8.(a) If $a + ib = 1/(x + iy)$, show that $(a^2 + b^2)(x^2 + y^2) = 1$.

$a + ib = (x - iy)/(x^2 + y^2)$

$a = x / (x^2 + y^2)$
$b = -y / (x^2 + y^2)$

$a^2 + b^2 = (x^2 + y^2) / (x^2 + y^2)^2$

$\mathbf{(a^2 + b^2)(x^2 + y^2) = 1}$

9. A box contains 20 fuses, of which 5 are defective. If 3 of the fuses are selected at random and removed from the box in succession without replacement, what is the probability that all three fuses are defective?

Probability first defective = 5/20

Probability second defective = 4/19

Probability third defective = 3/18

Required probability

$= (5/20)(4/19)(3/18)$
$= 1/114$

10. Find the volume V of a triangular pyramid ABCD with vertices A(2, −1, 1), B(5, 5, 4), C(3, 2, 1) and D(4, 1, 3).

AB = (3, 6, 3)

AC = (1, 3, 0)

AD = (2, 2, 2)

Volume

V = 1/6 | AB · (AC × AD) |

AC × AD =

| i j k |

| 1 3 0 |

| 2 2 2 |

= (6 − 0)i − (2 − 0)j + (2 − 6)k

= (6, −2, −4)

AB · (AC × AD)

= (3)(6) + (6)(−2) + (3)(−4)

= 18 − 12 − 12

= −6

V = 1/6 × 6

V = 1

11. Determine three geometric progression such that their sum is 52 given the condition that the sum of their product taken in pairs is 624.

Let the terms be a, ar, ar^2

Sum

$a(1 + r + r^2) = 52$

Sum of products in pairs

$a^2(r + r^2 + r^3) = 624$

Divide

$a = 4$

$1 + r + r^2 = 13$

$r^2 + r - 12 = 0$

$r = 3$

Terms are

4, 12, 36

12. If one root of the equation $8x^2 + qx + 27 = 0$ is the square of the other, find the value of q and hence rewrite the equation.

Let roots be α and α^2

Sum of roots

$\alpha + \alpha^2 = -q/8$

Product of roots

$\alpha^3 = 27/8$

$\alpha = 3/2$

$\alpha + \alpha^2 = 3/2 + 9/4$

$= 15/4$

$-q/8 = 15/4$

$q = -30$

Equation

$8x^2 - 30x + 27 = 0$

13. (a) If $\log_{10}2 = 0.3010300$ and $\log_{10}3 = 0.4771213$, without using tables, find

$\log_{10} 50$ to 7 decimal places.

$\log_{10} 50$

$= \log_{10}(5 \times 10)$

$= \log_{10}5 + 1$

$\log_{10}5 = 1 - \log_{10}2$

$= 0.6989700$

$\log_{10}50 = 1.6989700$

(b) $\log_{10} 13.5$ to 7 decimal places.

$13.5 = 27/2$

$\log_{10} 13.5$

$= \log_{10}27 - \log_{10}2$

$= 3 \log_{10}3 - \log_{10}2$

$= 3(0.4771213) - 0.3010300$

$= 1.4313639 - 0.3010300$

$= 1.1303339$

14.(a) If $p + q = 5$ and $p^2 + q^2 = 19$, find the value of pq and write the equation in x whose roots are p and q.

$p^2 + q^2 = (p + q)^2 - 2pq$

$19 = 25 - 2pq$

$2pq = 6$

$pq = 3$

Equation

$x^2 - 5x + 3 = 0$

(b) A geometric series with common ratio 0.8 converges to the sum 250. Find the fourth term of the series.

$S_\infty = a / (1 - r)$

$250 = a / 0.2$

$a = 50$

Fourth term

$= ar^3$

$= 50 \times (0.8)^3$

$= 25.6$

15.(a) If $y = (\sin^{-1}x)^2$, show that $(1 - x^2)(dy/dx)^2 = 4y$.

$y = (\sin^{-1}x)^2$

$dy/dx = 2 \sin^{-1}x / \sqrt{(1 - x^2)}$

$(dy/dx)^2 = 4(\sin^{-1}x)^2 / (1 - x^2)$

$(1 - x^2)(dy/dx)^2 = 4y$

(b) Find the equations of the tangent and normal to the curve $y = (1 - x)(2 + x)$ at the point where $x = 2$.

$y = 2 - x - x^2$

$dy/dx = -1 - 2x$

At $x = 2$

$dy/dx = -5$

Point is $(2, -4)$

Tangent

$y + 4 = -5(x - 2)$

Normal slope = 1/5

Normal

$y + 4 = 1/5(x - 2)$

SECTION B (51 marks)

Answer three questions from this section.

16.(a) If $z + 1/z = -1$, prove that $z^5 + 1/z^5 = -1$ and find the value of $z^{11} + 1/z^{11}$.

Given

$z + 1/z = -1$

Square both sides

$(z + 1/z)^2 = 1$

$z^2 + 2 + 1/z^2 = 1$

$z^2 + 1/z^2 = -1$

Multiply the original equation by $z^2 + 1/z^2$

$(z + 1/z)(z^2 + 1/z^2) = (-1)(-1)$

$z^3 + 1/z^3 + z + 1/z = 1$

Substitute $z + 1/z = -1$

$z^3 + 1/z^3 - 1 = 1$

$z^3 + 1/z^3 = 2$

Now multiply

$(z^2 + 1/z^2)(z^3 + 1/z^3)$

$= z^5 + 1/z^5 + z + 1/z$

Left side

$(-1)(2) = -2$

$-2 = z^5 + 1/z^5 + (-1)$

$z^5 + 1/z^5 = -1$

To find $z^{11} + 1/z^{11}$

Notice the cycle

$z + 1/z = -1$

$z^2 + 1/z^2 = -1$

$z^3 + 1/z^3 = 2$

$z^4 + 1/z^4 = 1$

$z^5 + 1/z^5 = -1$

The pattern repeats every 6 powers

11 mod 6 = 5

Therefore

$z^{11} + 1/z^{11} = z^5 + 1/z^5 = -1$

(a) Prove that $\cos^6\theta = 1/32 (\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10)$.

Use the identity

$\cos^2\theta = (1 + \cos 2\theta)/2$

Raise both sides to the power 3

$\cos^6\theta = (1/8)(1 + \cos 2\theta)^3$

Expand

$(1 + \cos 2\theta)^3$

$= 1 + 3 \cos 2\theta + 3 \cos^2 2\theta + \cos^3 2\theta$

Use identities

$\cos^2 2\theta = (1 + \cos 4\theta)/2$

$\cos^3 2\theta = (3 \cos 2\theta + \cos 6\theta)/4$

Substitute

$\cos^6\theta = 1/8 \, [1 + 3 \cos 2\theta + 3/2 \, (1 + \cos 4\theta) + 1/4 \, (3 \cos 2\theta + \cos 6\theta)]$

Simplify

$\cos^6\theta = 1/32 \, (10 + 15 \cos 2\theta + 6 \cos 4\theta + \cos 6\theta)$

17.(a) To estimate the true mean μ weight of automobiles, a random sample of size 100 has mean 2030 kg and standard deviation 216 kg. Find the 95 percent confidence interval for μ.

Sample mean $\bar{x} = 2030$

Standard deviation $s = 216$

Sample size $n = 100$

Standard error

$s/\sqrt{n} = 216/10 = 21.6$

For 95 percent confidence

$z = 1.96$

Margin of error

$1.96 \times 21.6 = 42.34$

Confidence interval

$2030 - 42.34 = 1987.66$

$2030 + 42.34 = 2072.34$

Therefore

$1987.66 \leq \mu \leq 2072.34$

(b) Two students received standard deviation scores of 0.8 and −0.4. Their marks were 88 and 64 respectively. Find the mean and standard deviation.

Let mean = μ and standard deviation = σ

Using $z = (x - \mu)/\sigma$

For first student

$0.8 = (88 - \mu)/\sigma$

$88 - \mu = 0.8\sigma$

For second student

$-0.4 = (64 - \mu)/\sigma$

$64 - \mu = -0.4\sigma$

Subtract equations

$(88 - \mu) - (64 - \mu) = 0.8\sigma + 0.4\sigma$

$24 = 1.2\sigma$

$\sigma = 20$

Substitute

$88 - \mu = 16$

$\mu = 72$

18.(a) Find a general solution to the second order differential equation $XY'' = 2$.

Given

$x\, y'' = 2$

$y'' = 2/x$

Integrate

$y' = \int 2/x \, dx$

$y' = 2 \ln x + C_1$

Integrate again

$y = \int (2 \ln x + C_1) \, dx$

$y = 2(x \ln x - x) + C_1x + C_2$

(b) Solve for initial value problem

$(x^2 + 1)y' + (y^2 + 1) = 0$ given $y(0) = 1$.

Rearrange

$dy/(y^2 + 1) = - dx/(x^2 + 1)$

Integrate

$\int dy/(y^2 + 1) = - \int dx/(x^2 + 1)$

$\tan^{-1}y = - \tan^{-1}x + C$

Apply initial condition

$\tan^{-1}(1) = - \tan^{-1}(0) + C$

$\pi/4 = C$

Solution

$\tan^{-1}y + \tan^{-1}x = \pi/4$

(c) If $z = \sin^{-1}((x - y)/(x + y))$, prove that $x\, \partial z/\partial x + y\, \partial z/\partial y = 0$.

Let

$u = (x - y)/(x + y)$

$z = \sin^{-1}u$

$\partial z/\partial x = (1/\sqrt{(1 - u^2)})\, \partial u/\partial x$

$\partial z/\partial y = (1/\sqrt{(1 - u^2)})\, \partial u/\partial y$

Compute derivatives

$\partial u/\partial x = 2y/(x + y)^2$

$\partial u/\partial y = -2x/(x + y)^2$

Multiply

$x\, \partial z/\partial x + y\, \partial z/\partial y$

$= (1/\sqrt{(1 - u^2)})\, [\, x(2y)/(x + y)^2 - y(2x)/(x + y)^2\,]$

$= 0$

19.(a) Evaluate

$\int$ from 1 to 2 $\int$ from 0 to 3 $\int$ from 0 to 1 $(p^2 + q^2 - r^2)\, dp\, dq\, dr$.

Integrate with respect to p

$\int (p^2 + q^2 - r^2)\, dp$

$= p^3/3 + pq^2 - pr^2$

Limits 0 to 1

$= 1/3 + q^2 - r^2$

Integrate with respect to q

$\int_0^3 (1/3 + q^2 - r^2)\, dq$
$= q/3 + q^3/3 - q\, r^2$

At $q = 3$

$= 1 + 9 - 3r^2$
$= 10 - 3r^2$

Integrate with respect to r

$\int_1^2 (10 - 3r^2)\, dr$

$= 10r - r^3$

At limits

$= (20 - 8) - (10 - 1)$
$= 12 - 9$
$= 3$

(b) Using Wallis formula evaluate
$I_6 = \int_0^{(\pi/2)} \sin^6 x\, dx$.

Wallis formula

$I_n = (n - 1)/n\, I_{n-2}$

$I_6 = 5/6\ I_4$

$I_4 = 3/4\ I_2$

$I_2 = 1/2\ I_0$

$I_0 = \pi/2$

Substitute

$I_2 = \pi/4$

$I_4 = 3\pi/16$

$I_6 = 5\pi/32$

20.(a) Find the local extrema for $f(x, y) = x^3 + y^3 - 6xy$.

First partial derivatives

$f_x = 3x^2 - 6y$

$f_y = 3y^2 - 6x$

Set to zero

$3x^2 = 6y$

$3y^2 = 6x$

$y = x^2/2$

$x = y^2/2$

Substitute

$x = (x^2/2)^2 / 2$

$x = x^4/8$

$x(x^3 - 8) = 0$

$x = 0$ or $x = 2$

Corresponding points

(0, 0) and (2, 2)

Second derivatives

$f_{xx} = 6x$
$f_{yy} = 6y$
$f_{xy} = -6$

At (0, 0)

$D = (0)(0) - 36 = -36$
Saddle point

At (2, 2)

$D = (12)(12) - 36 = 108$
Minimum point