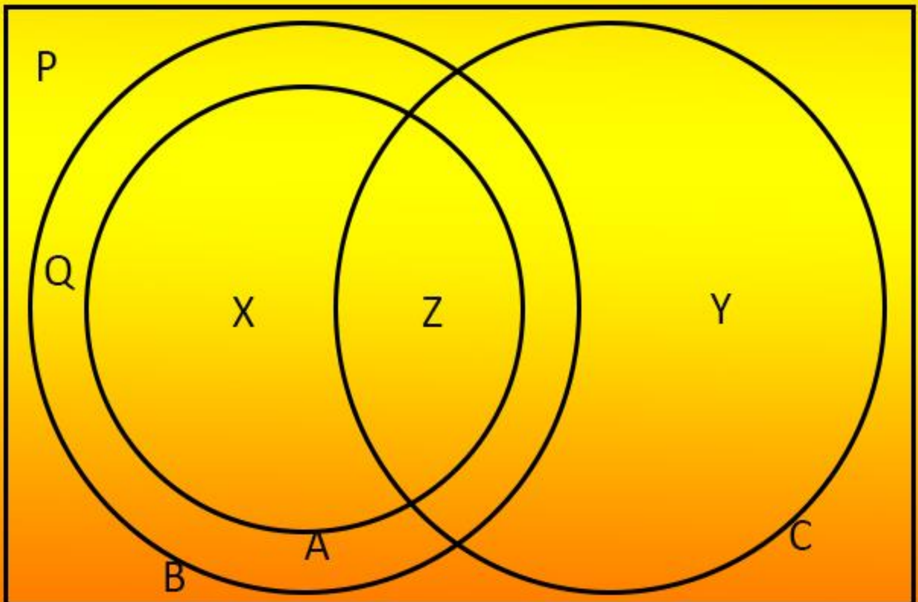
ADVANCED MATHEMATICS

SETS



Loibanguti, B.M

BSc. with Education - University of Dar Es Salaam, Tanzania

Author: Baraka Loibanguti

Editor: Baraka Loibanguti

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Cover Design: Baraka Loibanguti

Physical Address;
Baraka Loibanguti,
P.O Box 687,
Songea Town,
Ruvuma Tanzania.

Or

Baraka Loibanguti,
Njiro, Arusha,
Tanzania.

E-Mail: barakaloibanguti@gmail.com | info@jihudumie.com
Website: www.jihudumie.com
Tel: +255 62184 2525 | +254 735 748 429
Office Tel: +255 744 078 287 (Working Hours Only)

To my mom Faraja Alfayo Langoi

SETS

Chapter

2

Set is a well define collection of related objects. A set is denoted using capital letters and curl brackets. Thus A = {*elements*}

Example of sets

(a) Form five taking PCM in your school
(b) Male students in your class
(c) Leopards in Tarangire national park
(d) Whole numbers between 10 and 20
(e) An integer between 2 and 3

Elements of a set are members of a set or contains of a set. Consider set $A = \{a, b, c, d\}$, set A has elements *"a, b, c, d"*.

TYPES OF SETS

(a) Finite set – Is a set, which contain countable number of elements.
Example: $A = \{x : x \text{ is an integer and } 20 < x < 30\}$
Integers between 20 and 30 are 21, 22, 23, 24, 25, 26, 27, 28 and 29
Thus, $A = \{21, 22, 23, 24, 25, 26, 27, 28, 29\}$ has nine elements, therefore finite set.

(b) Infinite set – Is a set, which contain uncountable number of elements.
Example: $B = \{x : x \text{ is a positive whole number}\}$
Whole numbers are all non-decimal/fractional number.
Thus, $B = \{1, 2, 3, \ldots\}$ all positive whole numbers are uncountable positive, therefore the set has no end, thus makes set B an infinite set.

If set A become a set of real numbers, such that A = {x : x is a set of real numbers and 20 < x < 30}, then set A become infinite set because real numbers include tiny decimals.

(c) Empty set – Is a set, which has no element. It is denoted by sign ϕ or empty curl brackets { }
Example: A = {x : x is a prime number between 23 and 29}.
There is no a prime number between 23 and 29, therefore, A = { } = ϕ

(d) Singleton set – Is a set, which has one element. Example: A = {2} is a singleton set.

(e) Equivalent sets – These are two or more sets which have the same number of elements (cardinality). Suppose set A and B are equivalent sets, they are denoted as $A \equiv B$. Example, let $A = \{a, b, c\}$ and B = {1, 4, 7} then set A has three elements and so set B, therefore sets A and B are equivalents.

(f) Equal sets – Two or more sets are equal if they are equivalent and contain the same elements regardless of the order of the elements.
Example: Let $A = \{a, b, c\}$ and $B = \{c, a, b\}$ the two sets A and B has the same elements, and equal number of elements, therefore A and B are equal sets ($A = B$).

(g) Universal set – Is a big set, which contain all other given sets. It is denoted by sign μ or ξ. The universal set is a super set of all other given sets.
Example: Let μ= {x : x is an English alphabet}, A = {x : x is a vowel} and B = {x : x is a consonant}, both sets A and set B are in set μ, therefore μ is a universal set taking sets A and B in consideration.

REPRESENTATION OF SETS

Sets can be represented into two different ways

(a) Roster or list method

This method represents a set by listing some (for infinite set) or all (for finite set) elements of a set.
Example:

- If A = { all positive integers less than 10}
  By roster method is A = {1, 2, 3, 4, 5, 6, 7, 8, 9}
- If $B = \{x : x \in \text{integer}\}$
  By roster method is B = { … -3, -2, -1, … }

Real numbers are infinity given any interval, by roster method, we cannot list all numbers, and instead we use three dots (...) to show continuation. In any interval, let say $1 < x < 2$, there are infinite real numbers.

(b) Formula or set builder method

This method represents a set by formula. Examples of set representation in set-builder form are:-

- If $A = \{x : x \text{ is a positive integer less than } 10\}$
- If $B = \{x : x \in N, 11 < x \leq 20\}$
- If $C = \{x : x \in R, 11 < x \leq 12\}$

CARDINALITY OF A SET

Cardinality of a set is the number of elements of a set. Suppose set A is given, its cardinality is denoted by $n(A)$ or $|A|$

Example 1

Given $A = \{2, 3, 5, 6, 8, 9, 10, 13\}$ find the cardinality of set A

Solution

Cardinality of set A is the total element in set A, thus $|A| = n(A) = 8$

SUBSETS OF A SET

Let consider two sets A and B. If set A is a part of set B, such that all elements of set A are contained in set B, set A is called a subset of set B. Subset is denoted by $\subseteq$. If ($A \subseteq B$ and $A \neq B$) A is a proper subset of B and is written as $A \subset B$, which means A is contained in B, but A is not all in B (strict subset or proper subset).

Note that

(a) If $A \subseteq B$ and $B \subseteq A$ then $A = B$

(b) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Of course it is possible to have two sets, A and B, where neither is a subset of the other. Then A and B may share some elements, or no elements. In fact, for any given sets A and B, exactly one of the following will be true:-

(a) $A = B$ (set A equals to set B, thus $A \subseteq B$ and $B \subseteq A$)

(b) $A \subset B$ or $B \subset A$

(c) A and B share elements, but neither is a subset of the other
(d) A and B have no common elements, thus disjoint sets.

Theorems of subsets

(a) The empty set is a subset of any set
(b) Every set is a subset of itself
(c) Number of subsets of a set is given by the formula, $S = 2^n$ where n is the cardinality of a set.

Example 2

List all subsets of set A = {a, b, c}

Solution

First, determine how many subsets are in set A by using the formula $S = 2^n$

$\text{Subsets} = 2^3 = 8$, therefore there are 8 subsets in set A. When listing we start with empty set and we end with the set itself (refer the theorems above)

S = { }, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}.

Example 3

How many subsets are in set B = { 3, 4, 6, 7, 9}

Solution

Number of subsets is given by $S = 2^n$

Number of subsets in set B is $S = 2^5 = 32$ subsets

BASIC OPERATION OF SETS

1. UNION OF SETS [$A \text{ or } B = A \cup B$]

Let consider sets A and B, the union of two or more sets is denoted by $\cup$. The union of set A and B is written as $A \cup B$.

Definition

The union of two or more sets is a new set formed by putting together all elements, which are in individual sets without repeating elements.

Suppose, A = {2, 3, 5, 6, 8, 9} and B = {1, 3, 5, 7, 8} then $A \cup B$ = { 1, 2, 3, 5, 6, 7, 8, 9}.

2. INTERSECTION OF SETS [A and $B = A \cap B$]

The intersection is denoted by $\cap$. The intersection of set A and B is written as $A \cap B$.

Definition

The intersection of two or more sets is a new set formed by using the common elements of sets without repetition.

Suppose, A = {2, 3, 5, 6, 8, 9, 13} and B = {1, 3, 5, 7, 8}, then $A \cap B$ = {3, 5, 8}

3. COMPLEMENT OF SETS [$not\ A = A'$]

Consider the universal set μ, and set A. Set A complement is written as A' or A^c which is defined as the set of all elements which are in universal set, μ, but there are **not** in set A.

Example 4

Consider sets μ = {x : x is an integer, and 1 ≤ x ≤ 12}, A = {2, 4, 6, 8, 10, 12} and B = {1, 2, 4, 6, 7, 9}

Find

(a) $A \cap B$ (b) $A \cup B$ (c) $A \cup B'$ (d) $A' \cap B'$

(e) $A' \cup B'$ (f) Verify that $(A \cap B)' = A' \cup B'$

Solution

Given μ = {x : x is an integer, and 1 ≤ x ≤ 12}, A = {2, 4, 6, 8, 10, 12} and B = {1, 2, 4, 6, 7, 9}

(a) $A \cap B$ = {2, 4, 6}.

(b) $A \cup B$ = { 1, 2, 4, 6, 7, 8, 9, 10, 12}.

(c) Find first B' this is B' = { 3, 5, 8, 10, 11, 12, 13} therefore,
$A \cup B'$ = {2, 3, 4, 5, 6, 8, 10, 11, 12, 13}.

(d) A' = {1, 3, 5, 7, 9, 11, 13} therefore, $A' \cap B'$ = {3, 5, 11, 13}

(e) Using the above data, $A' \cup B'$ = { 1, 3, 5, 7, 8, 9, 10, 11, 12, 13}

(f) From (e) above $A' \cup B'$ = { 1, 3, 5, 7, 8, 9, 10, 11, 12, 13} and

$(A \cap B)'$ = {1, 3, 5, 7, 8, 9, 10, 11, 12, 13}, therefore $A' \cup B' = (A \cap B)'$ verified.

Example 5

Given that A = {(x,y): 2x + y = 5} and B = {(x,y): x + 3y = 5} find $A \cap B$

Solution

It is clear that both sets A and B are linear lines, therefore to get the intersection of these lines we need to solve the lines simultaneously, (by elimination method)

$$\begin{array}{c|l} 1 & 2x + y = 5 \\ 2 & x + 3y = 5 \end{array} \Rightarrow - \begin{array}{|l} 2x + y = 5 \\ 2x + 6y = 10 \end{array}$$

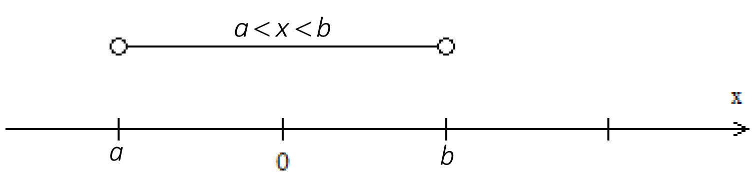
$-5y = -5 \Rightarrow y = 1$, using equation (ii) $x + 3y = 5 \Rightarrow x = 5 - 3 \Rightarrow x = 2$ therefore $A \cap B$ = {(x,y): x = 2 and y = 1}.

INEQUALITY PROBLEMS IN SETS

Inequalities in sets may have finite or infinite solution(s). The sign < or > are called exclusive signs which means the values/numbers pointed by these signs are not includes in a set/relation, these points are shown by an open small circle like, o . Using brackets the signs < or > is denoted by ( ). The signs ≤ or ≥ are called inclusive signs which means the values/numbers pointed by these signs are included points in a set/relation and they are denoted by a full shaded small circle like, ● . Using brackets the signs ≤ or ≥ is denoted as [ ].

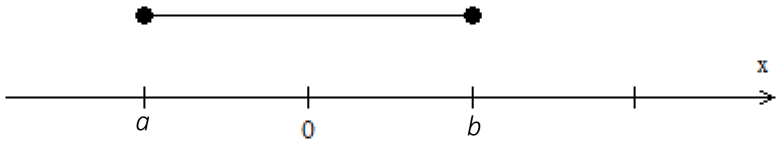
Brackets representation of inequality signs

(a) The round brackets shows open intervals (exclusive points), thus $(a,b) = a < x < b$



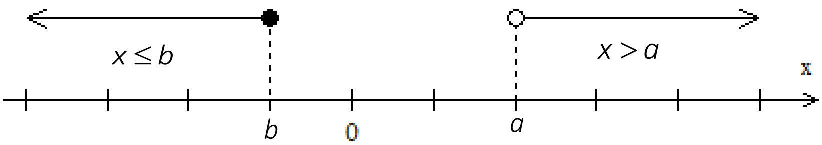
(b) The square brackets shows the closed intervals (included points), thus $[a,b] = a \le x \le b$

$$a \le x \le b$$



(c) One sided open and one sided closed intervals (half-opened), $[a,b) = a \le x < b$ or $(a,b] = a < x \le b$

Suppose the set A is defined as $x \le b$ and $x > a$ where a and b are real numbers, on a number line the solution is as shown below

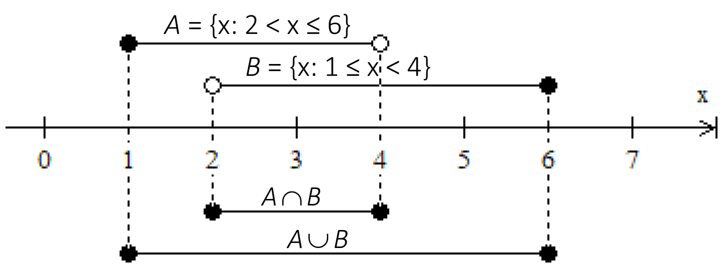


Example 6

If A = {x: 2 < x ≤ 6} and B = {x: 1 ≤ x < 4} show (a) $A \cap B$ (b) $A \cup B$ on a number line.

Solution

Draw a number line to represent sets A and B on the same straight line



From the number line above:-

(a) $A \cap B$ = {x: 2 ≤ x ≤ 4} (b) $A \cup B$ = {x: 1 ≤ x ≤ 6}

SIMPLIFICATION OF SETS EXPRESSIONS

Laws of algebra of sets

Some sets operations can be simplified and written in a simple form. To help simplifying sets algebra, the following laws are used.

(a) Idempotent law

Let A be a set, then

(a) $A \cup A = A$

(b) $A \cap A = A$

(b) Commutative law

Let A and B be any two sets, then

(a) $A \cup B = B \cup A$

(b) $A \cap B = B \cap A$

(c) Associative law

Let A, B and C be sets, then

(a) $A \cup (B \cup C) = (A \cup B) \cup C$

(b) $A \cap (B \cap C) = (A \cap B) \cap C$

(d) Distributive law

Let A, B and C be sets, then

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(e) De Morgan's law

Let A and B be two sets then

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

(f) Identity law

Let A, μ and ϕ be sets, where ϕ is an empty set and μ is an universal set, then

(a) $A \cap \phi = \phi$ (b) $A \cup \phi = A$

(c) $A \cap \mu = A$ (d) $A \cup \mu = \mu$

(g) Complement law

(a) $A \cap A' = \phi$ (b) $A \cup A' = \mu$

(c) $(A')' = A$ (d) $\mu' = \phi$

(d) $\phi' = \mu$

Example 7

Prove that $A\cap(B\cup C)=(A\cap B)\cup(B\cap C)$

Solution

Given $A\cap(B\cup C)=(A\cap B)\cup(B\cap C)$

Let $x\in A\cap(B\cup C)\Leftrightarrow(x\in A)$ and $(x\in B\cup C)$

$\Leftrightarrow(x\in A)$ and [$x\in B$ or $x\in C$]

$\Leftrightarrow$ [x $\in A$ and x $\in B$] or [x $\in A$ and x $\in B$]

$\Leftrightarrow$ (x $\in$ (A and B) or x $\in$ (A and C)

$\Leftrightarrow$ x $\in(A\cap B)$ or x $\in(A\cap C)$

This proof can also be done using Venn diagram

Example 8

Simplify using laws of algebra of sets $[A\cap(A'\cup B)]\cup[B\cap(A\cup B)']$

Solution

Given $[A\cap(A'\cup B)]\cup[B\cap(A\cup B)']$

$[(A\cap A')\cup(A\cap B)]\cup[(B\cap A)\cup(B\cap B')]$ - Distributive law

$[\varphi\cup(A\cap B)]\cup[(A\cap B)\cup\varphi]$ - Identity law

$[A\cap B]\cup[A\cap B]$ - Idempotent law

$A\cap B$

Example 9

Simplify the following using set properties $[A\cap(A\cup B)]\cup[B'\cap(A'\cup B')]$

Solution

Given $[A\cap(A\cup B)]\cup[B'\cap(A'\cup B')]$

$[(A\cup\varphi)\cap(A\cup B)]\cup[(B'\cup\varphi)\cap(A'\cup B')]$ - Identity law

$[A\cup(\varphi\cap B)]\cup[B'\cup(\varphi\cap A')]$ - Distributive law

$[A\cup\varphi]\cup[B'\cup\varphi]$ - Identity law

$A\cup B'$

Example 10

Using laws of algebra of sets simplify $(A-B)\cap(B-A)$

Solution

Given $(A-B)\cap(B-A)$

$(A\cap B')\cap(B\cap A')$ - Difference law

$(A\cap A')\cap(B'\cap B)$ - Commutative law

$\phi\cap\phi$ - Identity law

ϕ (empty set)

Therefore $(A-B)\cap(B-A)=\varphi$

Note that: $A-B$ mean A only and $B-A$ mean B only, thus $(A-B)\cap(B-A)$ is empty set ($\phi=\varphi$)

Example 11

Use the basic properties of sets operation simplify $(A\cap B')\cup(A\cup B')$

Solution

Given $(A\cap B')\cup(A\cup B')$

$(A\cap B')\cup(A\cup B')$

$[(A\cap B')\cup A]\cup B'$ - Associative law

$[(A\cap B')\cup(\mu\cap A)]\cup B'$ - Distributive law

$[A\cap(B'\cup\mu)]\cup B'$ - Identity law

$[A\cap\mu]\cup B'$ - Identity law

$A \cup B'$

EXERCISE 1 – SETS

Use laws of algebra of sets to simplify the following sets expressions

1. $(A \cap B') \cup (A' \cap B) \cup (A \cap B)$
2. $(A - B) \cup A$
3. $(A - B) - (A \cup B')$
4. $(A \cap B)' \cup [(A' - B') \cap (A/B')]$
5. $(A \cap B') \cup (A \cup B)$
6. $(A' \cup B')' \cup (A' \cup B)'$

   Prove the following identities

7. Show that $[A \cap (A' \cup B)] \cup [B \cap (A' \cup B)] = B$
8. Prove that $(A' \cup B) \cap (A' \cup B') = A'$
9. Show that $(A \cap B) \cup (A - B) = A$
10. Show that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
11. Prove that $A \cap (A' \cup B) \cap (A' \cup B') = \phi$
12. Show on a real number line $A \cap B$ if A = {x: x>2} and B = {x: x < 4}
13. If A = {x: -3 < x < 3} and B = {x: -1 < x ≤ 4} show on separate number line (a) $A \cap B$ (b) A' (c) B'
14. Given μ = {x: x ∈ R}, A = {x: -5 < x ≤ 0}, B = {x: x ≥ -3} and C = {x: 3 ≤ x} find
    (a) $A \cup B \cup C$
    (b) $A - B$
15. If μ = {x: -4 ≤ x < 11}, A = {x: 2 ≤ x ≤ 8}, B = {x: 1 ≤ x < 8} and C = {x: 3 < x < 10}
    (a) Is set A a subset of set B? Show by listing
    (b) Find $(A \cup C) \cap B$
    (c) Find $(A' \cap B') \cup C'$
16. Given A = {positive even numbers less than 10} and B = {positive prime numbers less than 10} show that $A \cap B \neq \phi$ and state the reason.
17. Verify the distributive law, given: A = {4, 5}, B = {3, 6, 7} and C = {2, 3}.

18. Given that μ = {pupils in a school} is a universal set, B = {boys}, H = {hockey players}, F = {football players}, expression the following in words:
(i) $F \subset B$ (ii) $H \subset B'$ (iii) $F \cap H \neq \phi$
Express in set notation:
(iv) No boys play football (v) All pupils play either football or hockey.
19. Use set properties to show that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

CARTESIAN PRODUCT OF SETS

Let A and B be two non-empty sets, the Cartesian product is defined as
$A \times B = \{(x, y): x \in A \text{ and } y \in B\}$
The Cartesian product of sets is not commutative, that is $A \times B \neq B \times A$

Example 12

If $A = \{3, 5, 7\}$ and $B = \{a, b\}$ find (a) $A \times B$ (b) $B \times A$

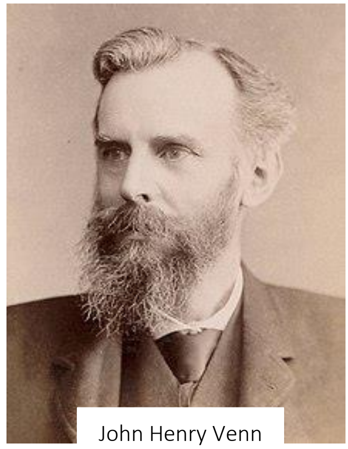
Solution

Multiplication method

(a) $A \times B = \{3, 5, 7\} \times \{a, b\}$
$A \times B = \{(3, a), (3, b), (5, a), (5, b), (7, a), (7, b)$
(b) $B \times A = \{a, b\} \times \{3, 5, 7\}$
$B \times A = \{(a, 3), (b, 3), (a, 5), (b, 5), (a, 7), (b, 7)\}$
This can also be done by tree diagram

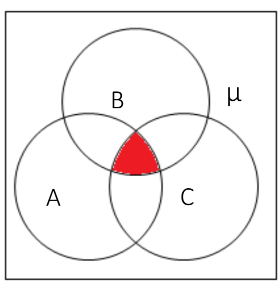
VENN DIAGRAM

John Henry Venn (4 August 1834 – 4 April 1923) was an English mathematician, logician and philosopher, came up with Venn Diagrams in 1881. They were a representation of the relation between sets using circles within circles. For example if we take three circles *A*, *B* and *C* which are all subsets of μ. The sections that are overlapping represent similar properties of the subsets whereas the independent areas were the individual properties of the sets. Venn diagrams can be applied in various problems however, they particularly aided in the Boolean logic named after the mathematician George Boole.

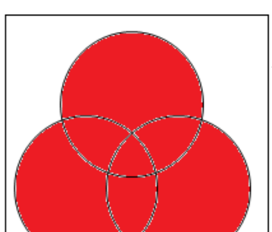

John Henry Venn

SET EXPRESSIONS AND THE CORRESPONDING REGION IN VENN DIAGRAMS

(a) Let *A, B* and *C* be three joint sets and let μ be a universal set, the region $A \cap B \cap C$ is the shaded region below



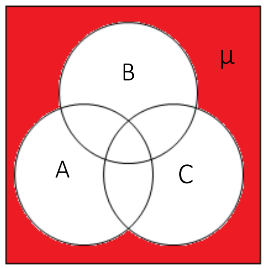
(b) Let *A*, *B* and *C* be three joint sets and let μ be a universal set, the region $A \cup B \cup C$ is the shaded region below



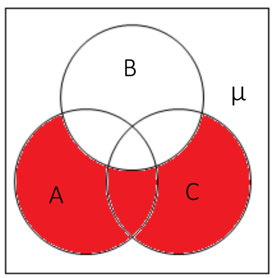
B μ

A C

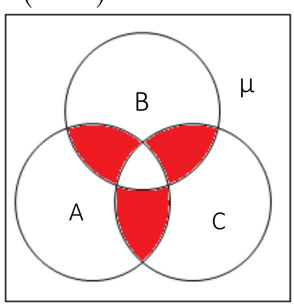
(c) Let A, B and C be three joint sets and let μ be a universal set, the region $(A \cup B \cup C)'$ is the shaded region below



(d) The region $(A \cup C) \cap B'$

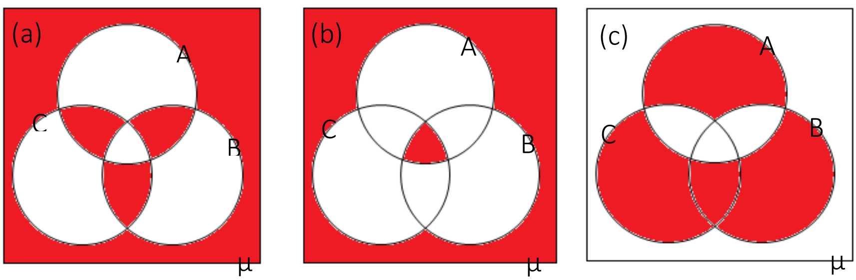


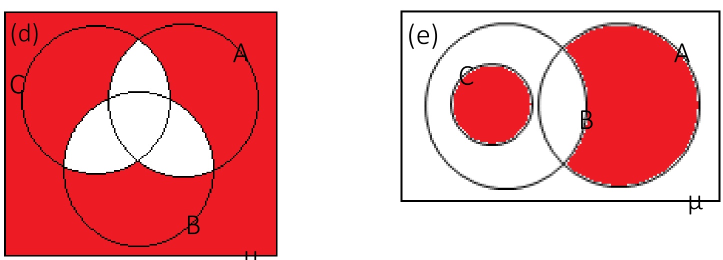
(e) The region $(A \cap B) \cup (B \cap C) \cup (A \cap C)$



Shade the following region in Venn diagrams

1. $(A \cap B) \cup C'$
2. $A' \cup C \cup B$
3. $A' \cap B \cup C'$
4. $(A' \cap B')' \cup C$
5. $A' \cup B \cap C$
6. $A' \cap B' \cup C'$
7. $A \cup B' \cap C'$
8. $(A \cap B') \cap C'$
9. $A \cup (B' \cap C)$
10. $(A \cap B) \cup (B \cap C')$
11. $(A \cup B)' \cap C'$
12. $A \cap (B' \cup C)$
13. $A' \cup (B \cap C)'$
14. $A' \cap (B \cap C)'$
15. Show that $(A \cap B) \cup C = (A \cup C) \cup (B \cup C)$ in a Venn diagram
16. Show that $(A \cup B \cup C)' = A' \cap B' \cap C'$ in a Venn diagram
17. Write the region represented by the following Venn diagrams





18. If P = {x, y, z} and Q = {a, b, c} find $P \times Q$ and $Q \times P$.
19. Let A = {1. 3, 5, 7, 9} and B = {2, 3, 5, 7, 11} find (a) $A \Delta B$ and (b) $B \Delta A$ verify if $A \Delta B = B \Delta A$

20. Shade in a Venn diagram the following regions (a) $A'\cap(B\cup C)$ (b) $A'\cap(C-B)$

21. If $n(C)=14$, $n(B\cap C)=7$, $n(A\cap C)=15$, $n(A\cup B)=20$, $n(A\cap B\cap C)=10$, $n(\mu)=40$ find by using a Venn diagram.

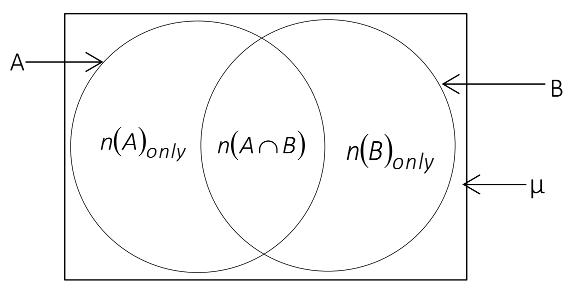
    (a) $n(A\cup B\cup C)$

    (b) $n(A\cap B\cap C)'$

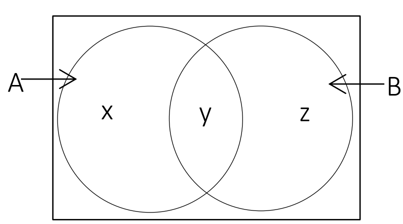
    (c) $n(A\cup B\cap C')$

NUMBER OF ELEMENTS OF TWO SETS

The number of elements of two sets is given by formula $n(A\cup B)=n(A)+n(B)-n(A\cap B)$ where *A* and *B* are joint sets, when *A* and *B* are disjoint sets the formula become $n(A\cup B)=n(A)+n(B)$ since set *A* and *B* has no common elements.



Consider the Venn diagram below



$n(A)=x+y$, $n(B)=y+z$ and $n(A\cap B)=y$

$n(A \cup B) = x + y + z$

$n(A \cup B) = (x + y) + (y + z) - y$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

When A and B are disjoint sets there is no common elements, thus

$A \cap B = \phi \Rightarrow n(A \cap B) = 0$

The formula become, $n(A \cup B) = n(A) + n(B)$

Other useful formulas

(a) $n(\mu) = n(A)' + n(A)$

(b) $n(\mu) = n(A \cap B) + n(A \cap B)'$

(c) $n(\mu) = n(A \cup B) + n(A \cup B)'$

(d) $n(A' \cap B) = n(A) - n(A \cap B)$

(e) $n(A \cap B') = n(B) - n(A \cap B)$

Example 13

If $n(A) = 70$, $n(B) = 60$ and $n(A \cap B) = 30$ find

(a) $n(A \cup B)$ (b) $n(A' \cap B)$ (c) $n(A \cap B')$

Solution

(a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A \cup B) = 70 + 60 - 30$

(b) $n(A' \cap B) = n(A) - n(A \cap B)$

$n(A' \cap B) = 70 - 30 = 40$

(c) $n(A \cap B') = n(B) - n(A \cap B)$

$n(A \cap B') = 60 - 30 = 30$

Example 14

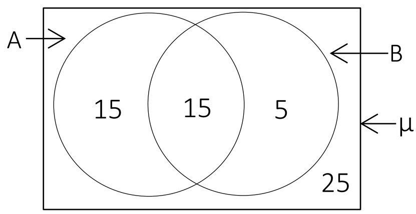
Given $n(\mu) = 60$, $n(A) = 30$, $n(B) = 20$ and $n(A \cup B) = 35$, represent this information in a Venn diagram and determine from the Venn diagram

(a) $n(A \cap B)$ (b) $n(A)'$ (c) $n(B)'$

(d) $n(A' \cup B)$ (e) $n(A \cap B)'$

Solution

Given $n(\mu)=60$, $n(A)=30$, $n(B)=20$ and $n(A \cup B)=35$,



(a) $n(A \cap B)=15$

(b) $n(A)'=25+5=30$

(c) $n(B)'=15+25=40$

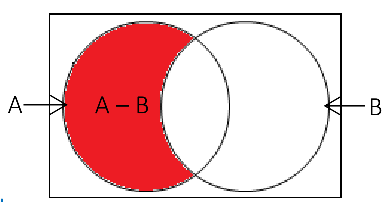
(d) $n(A' \cup B)=25+5=30$

(e) $n(A \cap B)'=15+5+25=45$

RELATIVE COMPLEMENT OF SETS

Let A and B be two non-empty sets, then the relative complement (difference of two sets) is denoted by $A-B$, which is defined as all elements which are in A and not in B, therefore $A-B=A \cap B'$

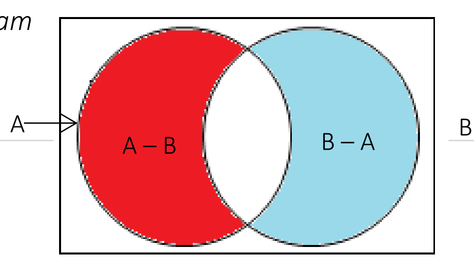
By the Venn diagram



Symmetric difference of sets

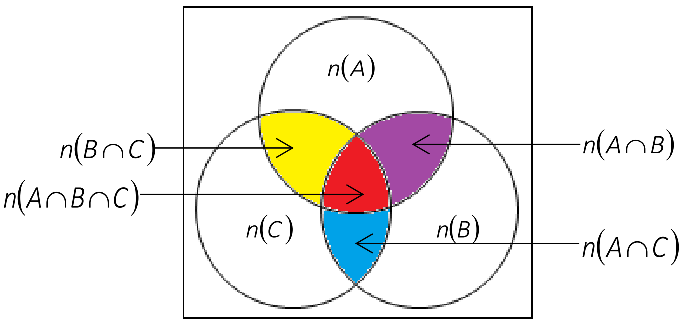
Let A and B be two sets. The symmetry difference is denoted by $A \Delta B$ or $A \oplus B$ which is defined as $A \oplus B = A \Delta B = (A-B) \cup (B-A)$. $A \Delta B$ is a set of all elements which are in a set A but not in set B or all elements which are in set B but not in set A.

By a Venn diagram



NUMBER OF ELEMENTS OF THREE SETS

Consider the Venn diagrams below



$$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(A\cap C)-n(B\cap C)+n(A\cap B\cap C)$$

Proof

From $n(A\cup B)=n(A)+n(B)-n(A\cap B)$

Let $P=(A\cup B)$

$n(P\cup C)=n(P)+n(C)-n(P\cap C)$ substitute $P=(A\cup B)$

$$n(A\cup B\cup C)=n(A\cup B)+n(C)-n((A\cup B)\cap C)$$

$$n(A\cup B\cup C)=n(A)+n(B)-n(A\cap B)+n(C)-[n(A)+n(B)-n(A\cap B)]\cap n(C)$$

$$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(A\cap C)-n(B\cap C)+n(A\cap B\cap C),$$

where A, B and C are joint sets

Example 15

If A, B and C are sets, A and C are disjoint sets, show that

$$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)$$

Solution

From the formula

$$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(A\cap C)-n(B\cap C)+n(A\cap B\cap C)$$

But $A \cap C = \varphi$ therefore $n(A \cap C) = 0$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) + n((A \cap C) \cap B)$ Associative law

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) + n(\varphi \cap B)$ Identity law

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) + n(B)$$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$$

SETS IN WORD PROBLEMS

Example 16

Spare parts produced by a certain factory were subjected to three types of defects A, B and C. A sample of 4000 items were inspected and it was found that 6.2% had defect A, 7.4% had defect B, 8.2% had defect C, 2.2% had defect A and B, 2.6% had defect B and C, 2.0% had defect A and C, and 1.2% had all three defects.

(a) Represent this information in a Venn diagram

Find the percentage of items, which had

(b) None of these defects

(c) At least one of these defects

(d) Not more than one defect

Solution

(a) Given

$$n(A) = 6.2\% = \frac{6.2}{100} \times 4000 = 248$$

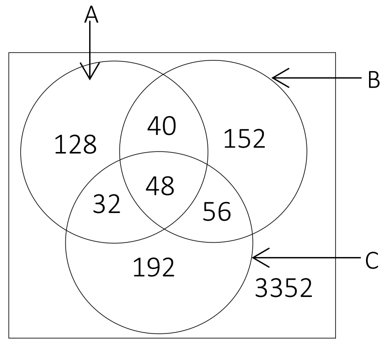
$$n(B) = 7.4\% = \frac{7.4}{100} \times 4000 = 296$$

$$n(C)=8.2\%=\frac{8.2}{100}\times 4000=328$$

$$n(A\cap B)=2.2\%=\frac{2.2}{100}\times 4000=88$$

$$n(B\cap C)=2.6\%=\frac{2.6}{100}\times 4000=104$$

$$n(A\cap C)=2.0\%=\frac{2}{100}\times 4000=80$$



$$n(A\cap B\cap C)=1.2\%=\frac{1.2}{100}\times 4000=48$$

(b) Non defect percent: $\frac{3352}{4000}\times 100\%=83.8\%$

(c) At least one of these defect: $\frac{648}{4000}\times 100\%=16.2\%$

(d) Not more than one defect: $\frac{3824}{4000}\times 100\%=95.6\%$

Example 17

A staff member at a large Engineering school was presenting data to show that the students were received a liberal education as well as Scientific one. “Looking at our record she said”, Out of one senior class of 500 students 350 are taking Mathematics, 250 are taking Physics, 196 are taking Physics and Mathematics, 87 are taking Physics and Foreign Language, 143 are taking Foreign Language and Mathematics and 36 are taking all of these. Present these data in a Venn diagram:-

(a) How many students are taking only one subject?

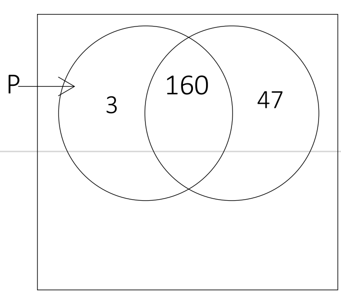
(b) How many students are taking only two subjects?

Solution

Let P be Physics, M be Mathematics and F be foreign language

Given that:-

$n(\mu)$ = 500



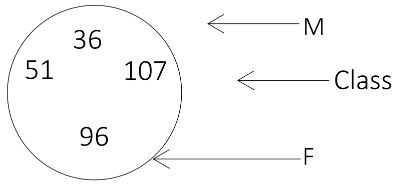
$n(P \cap M) = 196$

$n(M) = 350$

$n(P \cap F) = 87$

$n(F \cap M) = 143$

$n(P \cap F \cap M) = 36$



From the Venn diagram

(a) 146 students are taking only one subject.

(b) 318 students are taking only two subjects.

Example 18

A survey of 500 students taking one or more courses in Physics, Chemistry and Mathematics during one academic year revealed the following numbers of students in the indicated subjects:

Physics and Chemistry 83, Chemistry and Mathematics 63, Physics and Mathematics 217, Mathematics 295, Chemistry 186, Physics 329. Draw a Venn diagram to show the information given above and hence calculate the number of students taking Physics or Mathematics but not Chemistry.

Solution

Let P be Physics, M be Mathematics and C be Chemistry, then

$n(P \cap C) = 83$, $n(C \cap M) = 63$, $n(P \cap M) = 217$, $n(M) = 295$, $n(C) = 186$, $n(P) = 329$

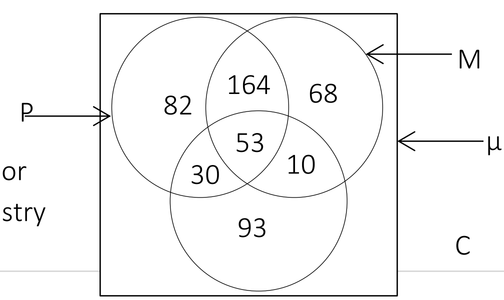
Using the set formula

$$n(P \cup M \cup C) = n(P) + n(M) + n(C) - n(P \cap M) - n(P \cap C) - n(M \cap C) + n(P \cap M \cap C)$$

$$500 = 329 + 295 + 186 - 83 - 63 - 217 + n(P \cap M \cap C)$$

$$500 = 447 + n(P \cap M \cap C)$$

$$n(P \cap M \cap C) = 53$$



From the Venn diagram

314 students taking Physics or Mathematics but not Chemistry

Example 19

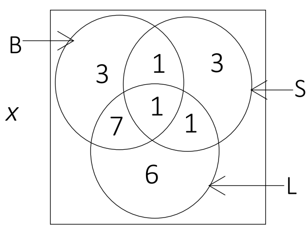
Twenty-four dogs are in a kennel. Twelve of the dogs are black, six of the dogs have short tails, and fifteen of the dogs have long hair. There is only one dog that is black with a short tail and long hair. Two of the dogs are black with short tails and do not have long hair. Two of the dogs have short tails and long hair but are not black. If all of the dogs in the kennel have at least one of the mentioned characteristics, how many dogs are black with long hair but do not have short tails?

Solution

Using a Venn diagram, let B be black dogs, S be short tails and L be long hair, then $n(B)=12$, $n(S)=6$, $n(L)=15$,

$n(B\cap S\cap L)=1$, $n(B\cap S\cap L')=n(B\cap S)=2$

$n(S\cap L\cap B')=n(S\cap L)=2$ and let $n(B\cap L)=x$



Using set formula

$24=12+6+15-2-2-(x-1)+1$

$24=31-x$ thus $x=7$

There are $3+7+6=16$ dogs black with long hair.

Example 20

In a group of 17 girls guides and 15 boys scouts, 22 play handball, 16 play basketball, 12 of the boys scout play handball, 11 of the boys scout play basketball, 10 of the boys scout play both games and 3 of the girls guide play neither of the games.

(a) How many girls in the group play both handball and basketball

(b) How many in the group play both basketball and handball

(c) How many in the group play

i) Handball only

ii) Basketball only

Solution

Let H be handball, B be basketball, G be girls

$n(H)=22=10+2+b+c$

$b+c=10$ (i)

$n(B)=16=10+1+a+b$

$a+b=5$ (ii)

$n(G)=17=a+b+c+3$

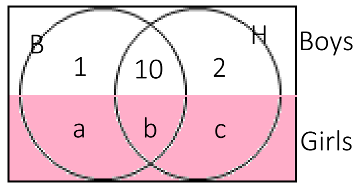
$a+b+c=14$

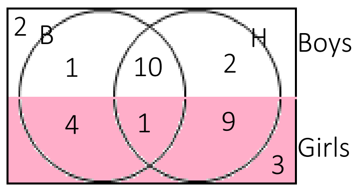
$5+c=14 \Rightarrow c=9$

$a+(b+c)=14$

$a+10=14 \Rightarrow a=4$ from $a+b=5$

$4+b=5 \Rightarrow b=1$





(a) 1 girl play both games.
(b) 11 girls and boys play both games.
(c) (i) 11 boys and girls plays handball only.
(ii) 5 boys and girls play basketball only.

Example 21

Out of 20 animals in a zoo, five animals eat grass, meat and bones, 6 animals eat grass and meat only, two animals eat grass and bones only and four eat meat and bones only. The number of animals eating one type of food only is divided equally between the three types of food

(a) Illustrate this information on a Venn diagram
(b) From the Venn diagram find the number of animals eating grass
(c) How many animals eat only one type of food?

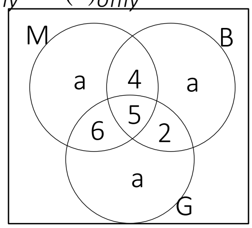
Solution

(a) Let G be grass, M be meat and B be bones

Given $n(G \cap M \cap B) = 5$, $n(G \cap M)_{only} = 6$, $n(G \cap B)_{only} = 2$,

$n(M \cap B)_{only} = 4$ and $n(M)_{only} = n(G)_{only} = n(B)_{only} = a$

Venn diagram



$3a + 2 + 5 + 4 + 6 = 20$

$3a + 17 = 20$ therefore $a = 1$

(b) 14 animals eat grass

(c) 3 animals eating one type of food

MISCELENOUS EXERCISE – SETS

1. At ABC Secondary School, thirty five students are taking Economics, the number of students who are taking Economic and Commerce is one quarter of all students. All those who are taking Accounts must also take Economics or Commerce or both. 27 students are taking Accounts. Those who are taking Economics and Accounts only are one seventh of those who are taking Economics, those who are taking Economics only are twice as many as those taking Commerce only and 15 are taking Accounts and Commerce only, if there are 72 students in this class, find the number of students taking:
    (a) Commerce and Economics but not Account
    (b) At least two subjects
    (c) Economics only
    (d) Commerce
    (e) Neither of the subjects
2. Sets *A*, *B*, and *C* are defined by $A = \{x: 2 \leq x \leq 20, x \in Z\}$, $B = \{x: x$ is a positive multiple of 2 less that $17\}$, $C = \{x: x$ is a factor of $5\}$ find $A \cap B \cap C$.
3. Each student in a class of 40 students studies at least one of the three subjects English, Mathematics and Economics. 18 study English, 22 Economics, 26 Mathematics, 5 takes English and Economics, 14 study Economics and Mathematics and 2 takes all three subjects. Find the number of students who study
    (a) English and Mathematics
    (b) English or Mathematics
    (c) English or mathematics but not Economics
4. The survey results show that out of 80 students from Arusha secondary school was found that 17 takes Mathematics, 20 takes Chemistry, 7 takes both Mathematics and Chemistry, 8 takes mathematics and Physics, those who are taking Physics and Chemistry are one fifth of those who do not take any subject, and 4 takes all three subjects, if no students takes Physics alone, Find the number of students that takes
    (a) Chemistry only

(b) Mathematics only
(c) Physics
(d) Physics and Chemistry but not mathematics
(e) Who study neither of the subjects
(f) Only one subject

5. (a) Using the basic properties of set operation simplify $(A-B)-(A\cup B')$

(b) An advertising agency finds that out of its 170 clients, 115 are television, 100 use radio, 130 use magazines, 75 use television and radio, 95 use radio and magazines, 85 use television and magazines and 75 use all three. Using Venn diagram determine how many use

(a) Radio only
(b) Television only
(c) Magazines only
(d) Television and magazines but not radio

6. Use laws of algebra of sets simplify $(A\cap B)'\cup[(A'-B')\cap(A/B')]$ hence draw the Venn diagram to illustrate the answer.

7. Using laws of algebra of sets, simplify the following expressions:

(a) $(A\cap B')\cup(A\cup B')$
(b) $(A\cap B')\cup(B'\cap A)\cup(A\cap B)$ represent the simplified answer of (a) and (b) in a Venn diagram

8. Given sets *A*, *B* and *C*, where *A* = {x ∈R: 2 < x ≤ 5}, *B* = {x ∈ R: x ≤ 2} and *C* = {x ∈ R: -2 ≤ x < 3} sketch on the number line $(A\cup B)\cap C$.

9. If μ is a universal set, $A\subset B$ and $B\subset C$ draw the Venn diagram representing the region simplify $(A'\cap B)\cup C'$.

10. In a certain examination students were given the option to choose at least one of the subjects he/she wants be examined in within Mathematics, Physics and chemistry. The results show that 15 took Mathematics, 20 took Physics, 21 took chemistry only, 4 took both mathematics and physics 8 took mathematics and chemistry and no students who took physics and chemistry without mathematics while only 2 took all three subjects.

(a) How many students in this option if every student had to choose at least one subjects?
(b) How many study chemistry

(c) How many took mathematics only
(d) How many took mathematics but not chemistry
(e) How many took physics only

11. In a class, 20 students takes Basic Mathematics, 15 takes Additional Mathematics and 10 takes physics. If the school regulation requires those who took Physics must take Additional mathematics and those who takes Additional Mathematics must take Basic Mathematics
    (a) Draw the Venn diagram to illustrate these information
    (b) How many students study Basic Mathematics, comment!
    (c) Find the number of students who study additional mathematics only
    (d) Find the probability that the student selected at random takes Additional mathematics and Physics
12. If $A = \{2, 3\}$, $B = \{1, 3, 5\}$, $C = \{3, 4, 7, 8\}$ and $D = \{1, 3, 5, 7\}$ find $(A \cap B) \times (C \cup D)$
13. Workers are grouped by their areas of expertise and are placed on at least one team. 30 are on the marketing team, 40 are on the sales team and 50 are on the visitors' team. 7 workers are on both the sales and visitors' team, 11 workers are on both marketing and visitor's team, none worked in marketing and sales only and 5 workers are on all three teams. How many workers are there in total? Hint: use the set formula.
14. Each of the 49 members in a high school class is required to sign up for a minimum of one and a maximum of three academic clubs. The three clubs to choose from are the TAKUKURU club, the history club, and the writing club. A total of 12 students sign up for the TAKUKURU club, 17 students for the history club, and 18 students for the writing club. If 6 students sign up for exactly two clubs, how many students sign up for all three clubs?
15. Of 20 Adults, *A, B* and *C* are organizations, *5* belong to *A*, 7 belong to *B*, and 9 belong to *C*. If 2 belong to all three organizations and 3 belong to exactly 2 organizations, how many belong to none of these organizations?
16. This semester, each of the 90 students in a certain class took at least one course from *A*, *B*, and *C*. If 60 students took *A*, 40 students took *B*, 20 students took *C*, and 5 students took all the three, how many students took exactly two courses?

17. In the city of Dar Es Salaam, 73 people own cats, dogs, or rabbits. If 30 people owned cats, 40 owned dogs, 10 owned rabbits, and 12 owned exactly two of the three types of pet, how many people owned all three?
18. When Professor Wang looked at the rosters for this term's classes, she saw that the roster for her economics class (E) had 26 names, the roster for her marketing class (M) had 28, and the roster for her statistics class (S) had 18. When she compared the rosters, she saw that E and M had 9 names in common, E and S had 7 names in common, and M and S had 10names in common. She also saw that 4 names were on all 3 rosters. If the rosters for Professor Wang's 3 classes are combined with no student's name listed more than once, how many names will be on the combined roster?
19. There are 50 employees in the office of *ABC* Company. Of these, 22 have taken an accounting course, 15 have taken a course in finance and 14 have taken a marketing course. Nine of the employees have taken exactly two of the courses and 1 employee has taken all three of the courses. How many of the 50 employees have taken none of the courses?
20. In a consumer survey, 85% of those surveyed liked at least one of three products: 1, 2, and 3. 50% of those asked liked product 1, 30% liked product 2, and 20% liked product 3. If 5% of the people in the survey liked all three of the products
    (a) What percentage of the survey participants liked more than one of the three products?
    (b) If there were 10,000 people in this survey, how many likes at most two products?
21. In a class of 50 students, 20 play Hockey, 15 play Cricket and 11 play Football. 7 play both Hockey and Cricket, 4 play Cricket and Football and 5 play Hockey and football. If 18 students do not play any of these given sports, how many students play exactly two of these sports?
22. If out of 200 shift workers, 60 are trained, 55 are trained and members of the workers union and 38 are members of workers union but not trained. How many are untrained shift workers who are not members of the union.
23. In Arusha city, 1,500,000 families were surveyed on bites, in which 24.8% of families buy super loaf bread, 29.3% buy family loaf and 23.5% buy Cakes, 10.3% of families buy super loaf and family loaf, 12% buy family loaf and cake,

11.5% of families buy super loaf and cake and 7% of families buy all three types of bites.

(a) Represent this information in a Venn diagram
(b) Find the number of families which buy super loaf
(c) Find the number of families which buy cakes and family loaf only
(d) Find the number of families which buy either cakes or family loaf
(e) How many families use other type of bites?

24. (a) Use laws of algebra of sets to simplify the following expressions: -

(i) $[(A-B)-B-A]'$ (ii) $(A\cup B)'\cap(A\cap B)'$ (iii) $A-(A'-B)$

(b) A certain class has 15 boys who like dancing, 5 who are handsome and 6 who are intelligent. Every handsome boy in the class like dancing, 2 boys are both handsome and intelligent and 3 boys like dancing and they are intelligent. Use a Venn diagram to present the information above and determine the number of boys who:-

(i) Like dancing but are not handsome.
(ii) Are just intelligent or they just like dancing.
(iii) The number of boys in the class.

25. Twenty-four people go on holidays. If 15 go swimming, 12 go fishing, and 6 do neither, how many go swimming and fishing? Draw a Venn diagram and fill in the number of people in all four region.

26. A class of 34 boys and 28 girls. In the class there are:-

45 students studying Kiswahili
33 students studying Mathematics
22 boys studying Kiswahili
17 boys studying Mathematics
10 boys studying both Mathematics and Kiswahili
2 girls studying neither Mathematics nor Kiswahili

How many students in the class are studying: -

(a) Both Mathematics and Kiswahili?
(b) Only one subject?
(c) Study neither of the mentioned subjects?

27. (a) Simplify $(A\cap B')\cup(A'\cap B)\cup(A\cap B)$

(b) An investigation of eating habits of 110 rabbits revealed that 50 rabbits eat rice, 43 eat maize, 45 eat banana, 12 eat rice and maize, 13 eat maize and banana, 15 eat banana and rice and 5 eat all types of food. Summarize the given information on a Venn diagram.

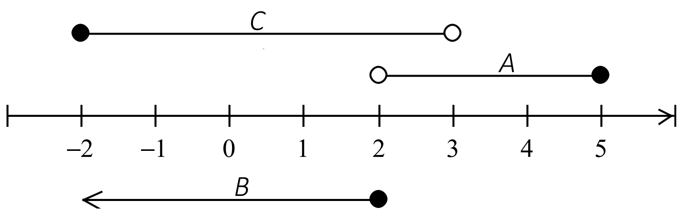
(c) Use the Venn diagram obtained in part (b) to find the number of rabbits which eat;
   (i) Only one type of food
   (ii) Banana and maize but not rice
   (iii) None of the above

EXERCISE 1 – SETS

1. μ
2. A
3. ϕ
4. $A' \cup B'$
5. A
6. A
12. $A \cap B = \{2 \le x \le 4\}$
13. (a) $A \cap B = \{x: -1 \le x \le 3\}$ (b) $A' = \{x: x \le -3 \text{ and } x \ge 3\}$ (c) $B' = \{x: x \le -1 \text{ and } x > 4\}$
14. (a) $A \cup B \cup C = \{x: x \in R\}$ (b) $A - B = \{x: -5 \le x \le -3\}$
15. (a) A is not a subset of B (b) $(A \cup C) \cap B = \{x: 1 \le x \le 8\}$ (c)
16.

MISCELLENEOUS EXRECISE – SETS

1. (a) 11 (b) 38 (c) 12 (d) 39 (e) 16
2. $A \cap B \cap C = \{2, 6, 10\}$
3. (a) 9 (b) 35 (c) 18
4. (a) 13 (b) 6 (c) 18 (d) 10 (e) 40 (f) 19
5. (a) ϕ (b) (i) 5 (ii) 30 (iii) 25 (iv) 10
6. $A' \cup B'$
7. (a) $A \cup B'$ (b) $A \cap B$
8. $(A \cup B) \cap C = C$



10. (a) 52 (b) 29 (c) 5 (d) 7 (e) 16
11. (b) 20 (c) 5 (d) 0.5
12. $(A \cap B) \times (C \cap D) = \{(3, 3), (3, 7)\}$
13. 102 14. 18 15. None
16. 35 17. 5 18. 50
19. 7
20. (a) 10% (b) 9,500
21. 10
22. 102
23. (b) 380,000 (c) 10,000 (d) 420,000 (e) 290,000
24. (a) (i) μ (ii) $A' - B$ (iii) A (b) (i) 10 (ii) 18 (ii) 18

25. 9
26. (a) 23 (b) 32 (c) 7

Advanced Mathematics
Sets

Sets is the second topic for the advanced matematics Tanzania syllabus. This book covers all necessary parts of the topic to help learners and facilitators. The exercise provided in this book has got the suggested answers at the end of the book. The writer also wrote other books from primary level to advanced level mathematics and Information and Communication Technology books, to get other books visit www.jihudumie.com and navigate to the library.

For inquiries info@jihudumie.com or +255 621 84 2525

Loibanguti, B.M

www.jihudumie.com